Listing Bit Strings

List all bit strings of length 3.

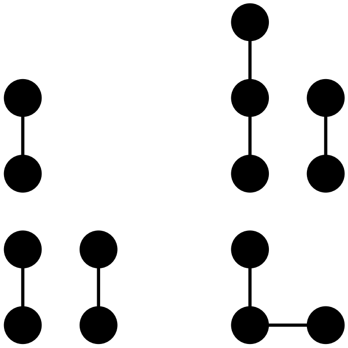
000, 001, 010, 011, 100, 101, 110, 111.

Now do it while only flipping one bit at a time!

Today: Finish graphs and talk about numbers.

Forests

A **forest** is an acyclic graph.

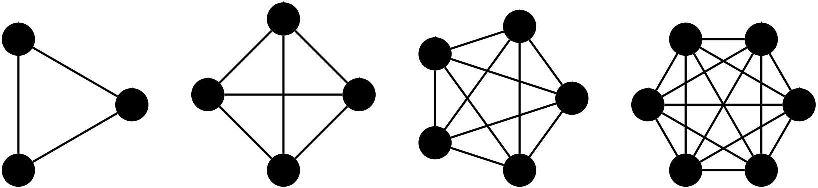


Each connected component of a forest is a tree.

How many connected components in this graph? 6.

Complete Graphs

The **complete graph** K_n has n vertices and *all* possible edges.



A **bipartite graph** has **left nodes** L and **right nodes** R.

- The vertex set is $V = L \cup R$.
- Left nodes are only allowed to connect to right nodes; right nodes are only allowed to connect to left nodes.

The **complete bipartite graph** $K_{m,n}$ has m left nodes, n right nodes, and *all* possible edges.

Edge Sparsity

How many edges does K_n have?

- Handshaking Lemma: $\sum_{v \in V} \deg v = 2|E|$.
- $\sum_{v \in V} \deg v = n(n-1)$.
- So $|E| = n(n-1)/2$.

Asymptotic notation from CS 61A/B: $|E| = \Theta(n^2)$.

For a tree on n vertices, $|E| = n-1 = \Theta(n)$.

The complete graph is called *dense*; trees are called *sparse*.

Planar Graphs Are Sparse

Theorem: For a connected planar graph with $|V| \geq 3$, we have $e \leq 3v - 6$.

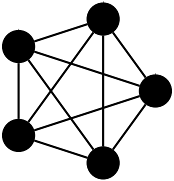
Proof.

- Each edge has two “sides”. So, if we add up all of the sides, we get $2e$.
- Each face has at least three sides. So the total number of sides is at least $3f$.
- Thus, $2e \geq 3f$.
- Euler’s Formula: $v + f = e + 2$.
- Rearrange: $e \leq 3v - 6$. $\square$

If the graph has n vertices, then $|E| = \Theta(n)$. Like trees.

Planar graphs are sparse.

K_5 Is Not Planar



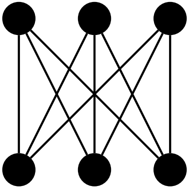
How many edges does K_5 have? 10.

- $e = 10$.
- $3v - 6 = 9$.

This violates $e \leq 3v - 6$ for planar graphs.

K_5 is not planar.

$K_{3,3}$ Is Not Planar



Consider $K_{3,3}$. Edges? 9. Vertices? 6. So $3v - 6 = 12$.

The previous proof fails. Make it stronger!

- The total number of sides is $2e$.
- Each face has at least three sides. Actually, at least four!
- In a bipartite graph, cycles are of even length.
- So, $2e \geq 4f$ and $v + f = e + 2$, so rearranging gives $e \leq 2v - 4$ for bipartite planar graphs.

Conclusion: $K_{3,3}$ is not planar.

Why K_5 and $K_{3,3}$?

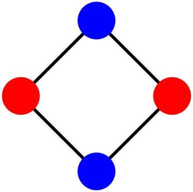
Why did we show that K_5 and $K_{3,3}$ are non-planar?

Kuratowski's Theorem: A graph is non-planar if and only if it "contains" K_5 or $K_{3,3}$.

- The word "contains" is tricky... do not worry about the details. Not important for the course.
- Content of theorem: essentially K_5 and $K_{3,3}$ are the only obstructions to non-planarity.

Graph Coloring

A **(vertex) coloring** of a graph G is an assignment of colors to vertices so that no two colors are joined by an edge.



Why do we care about graph coloring?

- Edges are used to encode *constraints*.
- Graph colorings can be used for scheduling, etc.

Coloring with Maximum Degree $+1$

Theorem. Let $d_{\max}$ be the maximum degree of any vertex in G. Then G can be colored with $d_{\max} + 1$ colors.

Proof.

- Use induction on $|V|$.
- For $|V| \geq 2$, remove a vertex v.
- Inductively color the resulting graph with $d_{\max} + 1$ colors.
- Add v back in.
- Since v has at most $d_{\max}$ neighbors which use at most $d_{\max}$ colors, use an unused color to color v. $\square$

For some types of graphs, this bound is very bad.

Bipartite Graphs Are 2-Colorable

Theorem: G is bipartite $\iff$ G can be 2-colored.

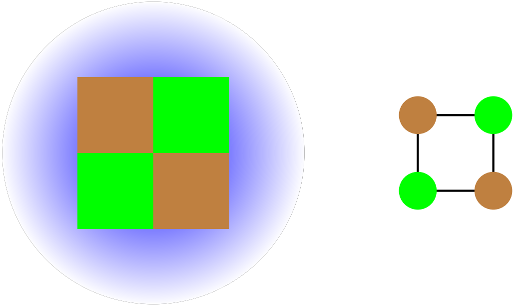
Proof.

- If G is bipartite with $V = L \cup R$, color vertices in L blue and vertices in R red.
- Conversely, suppose G is 2-colorable.
- In the 2-coloring of G, the red vertices have no edges between them, and similarly for blue vertices.
- So the graph is bipartite. □

Consider $K_{n,n}$. Then $d_{\max} + 1 = n + 1$, but it can be 2-colored.

Graph Coloring & Planarity

Consider a colored map and its planar dual:



(Ignore the infinite face.)

Coloring a map so no adjacent regions have the same color is equivalent to coloring a planar graph.

Four Color Theorem

Four Color Theorem: Every planar graph can be 4-colored.

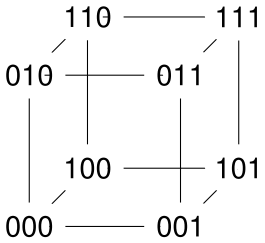
- The proof required a human to narrow down the cases, and a computer to exhaustively check the remaining cases.
- The proof has not yet been simplified to the point where a human can easily read over it.
- Note: K_5 requires 5 colors.

Hypercubes

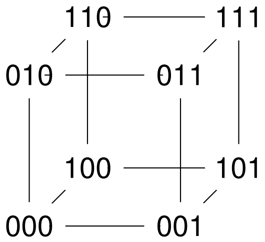
The **hypercube** of dimension *d*, Q_d, where *d* is a positive integer, has:

- vertices which are labeled by length-*d* bit strings, and
- an edge between two vertices if and only if they differ in exactly one bit.

Here is a picture of Q_3.



Hypercube Facts



The 0**-face** is the part of the hypercube whose vertices begin with 0. Similarly for the 1**-face.**

The 0-face is a lower-dimensional hypercube. Induction!

Number of vertices? 2^d.
Number of edges? $\sum_{v \in V} \deg v = d2^d$, so $|E| = d2^{d-1}$.

So for a hypercube with n vertices, $|E| = \Theta(n \log n)$.

Hypercubes Are Bipartite

Theorem: Hypercubes are 2-colorable.

Proof.

- Color all vertices with an even number of 0s blue and an odd number of 0s orange.
- Since each edge flips a bit, edges only connect vertices of different parity. □

Inductive Proof.

- Check the base case.
- Inductively color the 0-face.
- If $0x$ is a vertex colored blue, color the vertex $1x$ orange and if $0x$ is orange, color $1x$ blue. □

Hamiltonian Paths

Recall: List all bit strings of length 3, flipping one bit at a time.

A **Hamiltonian cycle** is a cycle that includes every vertex exactly once.

Listing the bit strings while flipping one bit at a time is exactly a Hamiltonian cycle on the hypercube.

Inductive construction:

- Length 1: 0, 1.
- Length 2: Length-1 sequence with 0s prepended. 00, 01.
  Length-1 sequence *backwards* with 1s prepended. 11, 10.
  Put it together: 00, 01, 11, 10.
- Length 3: 000, 001, 011, 010, 110, 111, 101, 100.

Hypercubes have Hamiltonian cycles.

Clock Mathematics

If it is 2:00 right now, what time is it in 24 hours? Still 2:00.

In the clock mathematics, the numbers *wrap around*: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1, 2, 3, . . .

We will do the same thing for bases other than 12.
Also, we will typically use the representatives $\{0, 1, \ldots, 11\}$ rather than $\{1, \ldots, 12\}$.

Question to ponder: What time will it be in $2^{1000000}$ hours from now? Can this even be computed?

Modular Equivalence

Let m be a positive integer.
For the next few lectures, m will be called the **modulus**.

Say that $x \equiv y \pmod{m}$ if $m \mid x - y$.
Read this as "x is equivalent to y, modulo m."

Examples: What numbers are equivalent to 0, modulo 6?

- $\ldots, -18, -12, -6, 0, 6, 12, 18, \ldots$.

In the "modulo 6" system, think of these numbers as the same.

Modular Equivalence: Addition, Multiplication

Theorem: If $a, b, c, d \in \mathbb{Z}$ with

$$a \equiv c \pmod{m} \qquad \text{and} \qquad b \equiv d \pmod{m},$$

then $a + b \equiv c + d \pmod{m}$ and $ab \equiv cd \pmod{m}$.

Addition and multiplication work as usual in modular arithmetic.

Proof.

- By definition, $m \mid a - c$ and $m \mid b - d$.
- So, $m \mid a + b - (c + d)$.
- Also $a = km + c$ and $b = \ell m + d$ for some $k, \ell \in \mathbb{Z}$.
- So, $ab = k\ell m^2 + dkm + c\ell m + cd$.
- Hence $m \mid ab - cd$. □

Representatives

Theorem: Each integer x is equivalent to a unique member of $\{0, 1, \ldots, m-1\}$ modulo m.

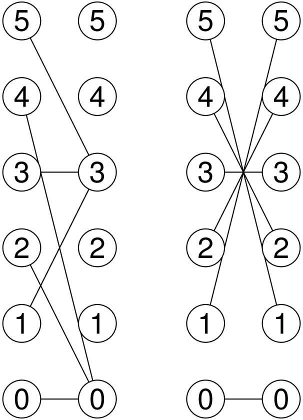
Proof.

- By Division Algorithm, $x = qm + r$ for some $q \in \mathbb{Z}$ and $r \in \{0, 1, \ldots, m-1\}$.
- Thus $m \mid x - r$, i.e., $x \equiv r \pmod{m}$.
- If $x \equiv r_1 \pmod{m}$ and $x \equiv r_2 \pmod{m}$, then (by subtracting) $r_1 - r_2 \equiv 0 \pmod{m}$.
- But this is impossible if $r_1, r_2 \in \{0, 1, \ldots, m-1\}$ are distinct. □

Now we can think of the numbers $\{0, 1, \ldots, m-1\}$ with addition and multiplication (modulo m) as a number system.
This system is usually called $\mathbb{Z}/m\mathbb{Z}$.

Multiplication in Modular Arithmetic

Modulo 6:



Left: Going from left to right is multiplication by 3.

Right: Going from left to right is multiplication by 5.

Bijections

A function $f : A \to B$ is:

- **injective** (or **one-to-one**) if for $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$ (different inputs mapped to different outputs);
- **surjective** (or **onto**) if for every $y \in B$, there is an $x \in A$ with $f(x) = y$ (every element of B is hit);
- **bijective** if it is both injective and surjective.

A bijection is like relabeling the elements of A.

Consider the map "multiplication by a, modulo m". That is, $f(x) := ax \bmod m$.

When is this map bijective?

Greatest Common Divisor

For two integers $a, b \in \mathbb{Z}$, the **greatest common divisor (GCD)** of a and b is the largest number that divides both a and b.

Fact: Any common divisor of a and b also divides $\gcd(a, b)$.

(Proof: Next time!)

Existence of Multiplicative Inverses

Theorem: $f(x) = ax \bmod m$ is bijective if and only if $\gcd(a, m) = 1$.

For $a \in \mathbb{Z}/m\mathbb{Z}$, a **multiplicative inverse** x is an element of $\mathbb{Z}/m\mathbb{Z}$ for which $ax \equiv 1 \pmod{m}$.

Corollary: For all $a \in \mathbb{Z}/m\mathbb{Z}$, a has a multiplicative inverse (necessarily unique) if and only if $\gcd(a, m) = 1$.

(Proof: Next time!)

Summary

Graphs.

- Consequences of Euler's Formula: non-planarity of K_5 and $K_{3,3}$; planar graphs are sparse.
- Types of graphs: forests, hypercubes.
- Graph colorings: $\leq d_{\max} + 1$ for general graphs, 2 for bipartite graphs.
- Hypercubes have Hamiltonian cycles.

Modular arithmetic.

- $a \equiv b \pmod{m}$ if $m \mid a - b$.
- Each number modulo m has a representative in $\{0, 1, \ldots, m-1\}$.
- Injections, surjections, bijections...
- a has a multiplicative inverse modulo m if and only if $\gcd(a, m) = 1$.