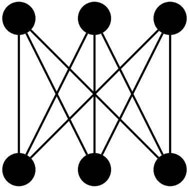
A Graph Puzzle



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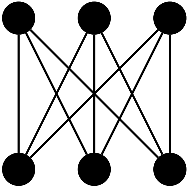

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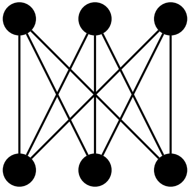

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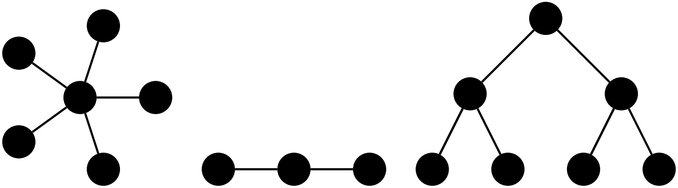
Today: We study special graphs (trees and planar graphs).

Trees

A **tree** is a connected acyclic graph.



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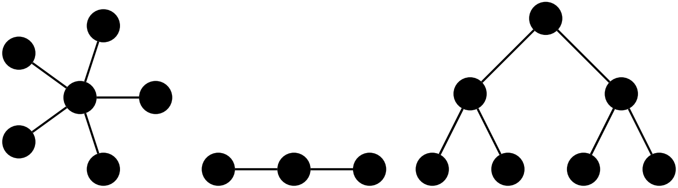

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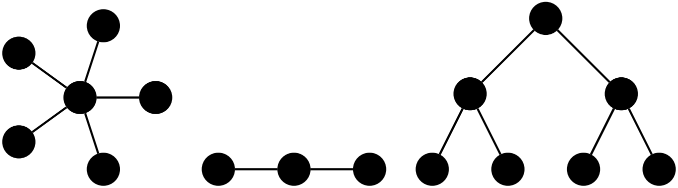

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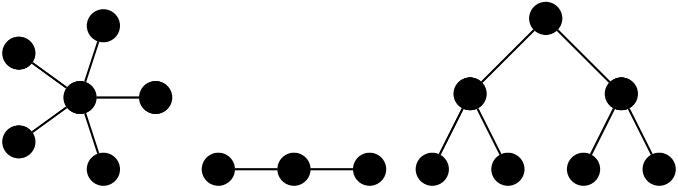

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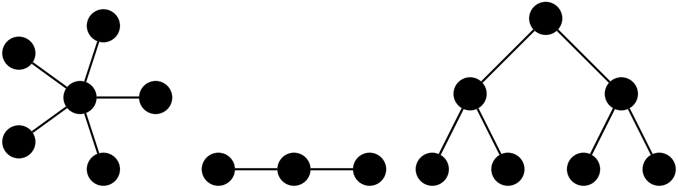

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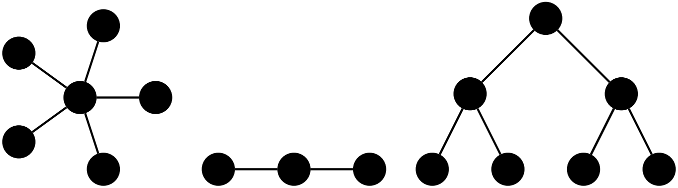

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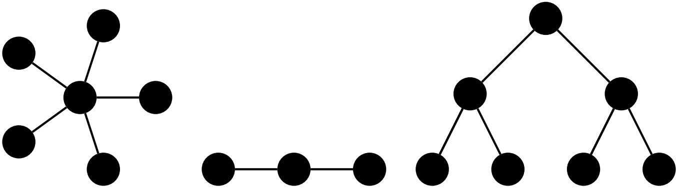

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We need to remove leaves to do induction.

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- Thus our tree was also connected and acyclic (leaves cannot be in cycles). $\square$

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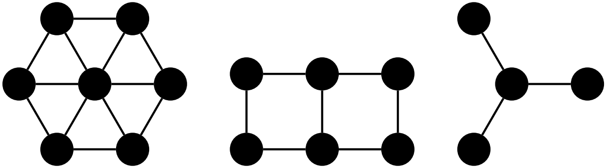
- Zoom in on v. Since v has finitely many attached edges, there must be room to draw $\{u, v\}$ without crossings. $\square$

Euler’s Formula



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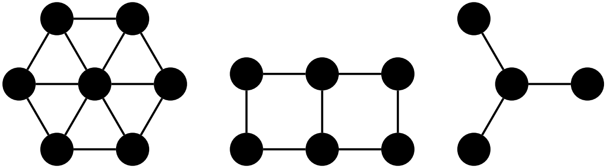

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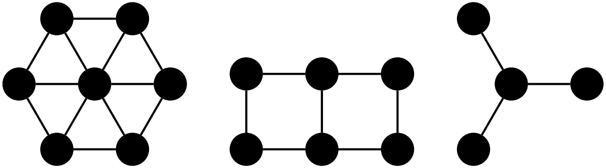

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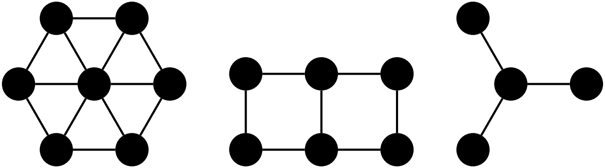

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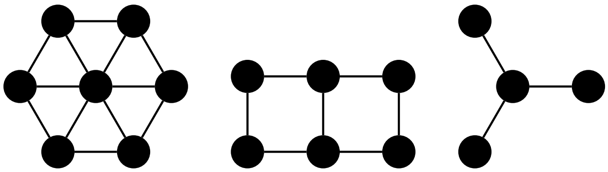

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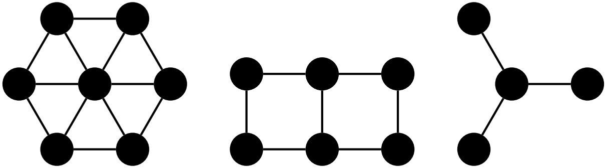

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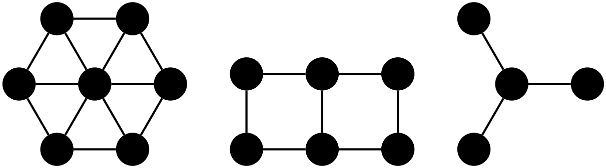

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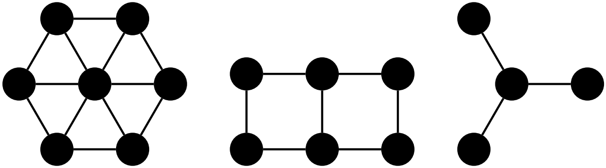

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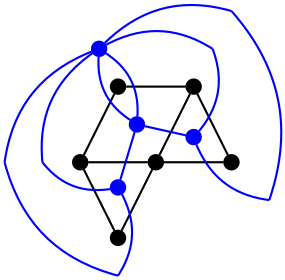
Trees: $|V| + 1 = |V| - 1 + 2$.

Planar Duality

Given a connected planar graph G, we define the **dual planar graph** G^*:



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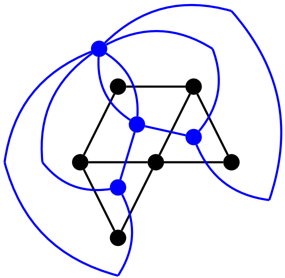

- Each **face** in G becomes a **vertex** in G^*.

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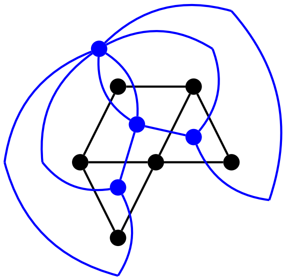
Given a connected planar graph G, we define the **dual planar graph** G^*:



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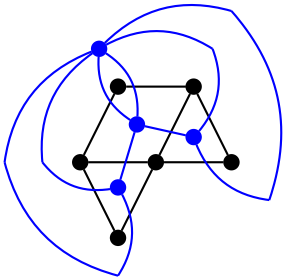

- Each **face** in G becomes a **vertex** in G^*.
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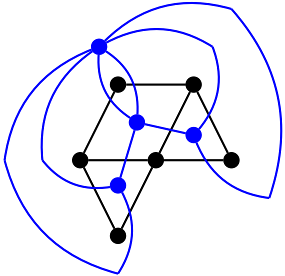
- Technically, we should say *a* dual, instead of *the* dual—there may be multiple planar duals for G.

The Dual of the Dual

If G^* is a planar dual of G, then G is a planar dual of G^*.



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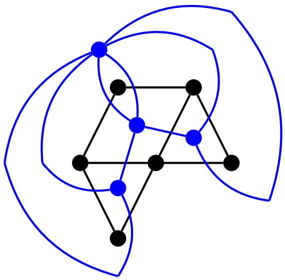

Just stare at it!

Proof of the Dual of the Dual

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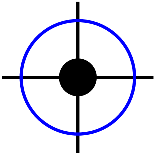
- Look at a vertex.



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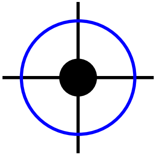

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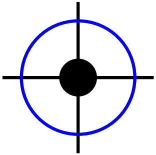

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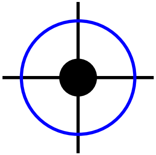

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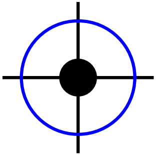
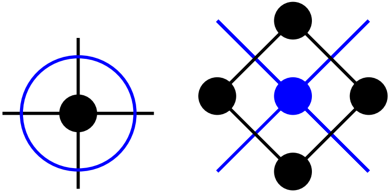


In G^*, the lines going through the edges incident to the vertex define a face.

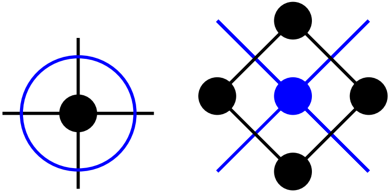
- The face encloses the vertex.
- Thus, the vertices of G correspond to the faces of G^*.
- We already know that the edges of G and G^* correspond to each other. □

Cycle-Cut Duality



The previous argument is a special case of cycle-cut duality.

Cycle-Cut Duality



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A **cut** is a set of edges which, if removed, separates a set of vertices from the rest of the vertices.

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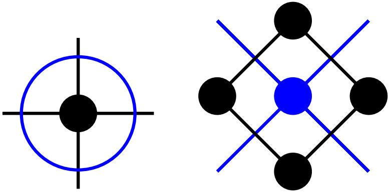

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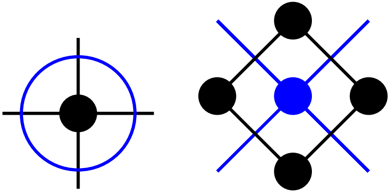

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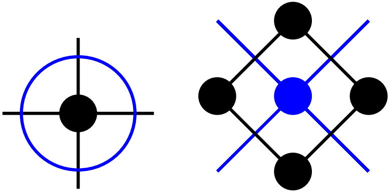

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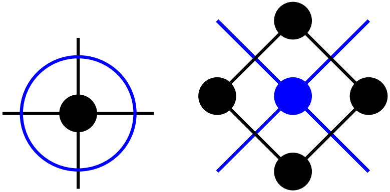

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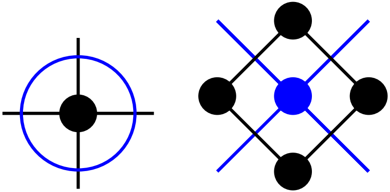

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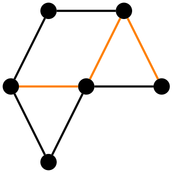
- Since G is a dual of G^*, then a simple cut in G corresponds to a cycle in G^*.

Cuts & Connectedness



Consider a set of edges with no cuts.

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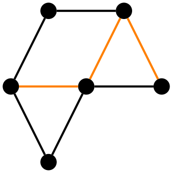

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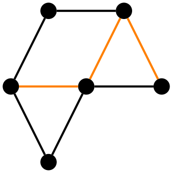

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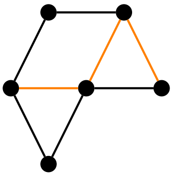

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If a set of edges connects their vertices, then the remaining edges must not have any cuts for these vertices.

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Start with a connected planar graph G. Find a **spanning tree**: a set of edges which form a tree in G.

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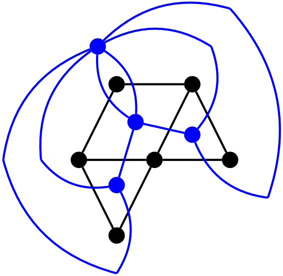

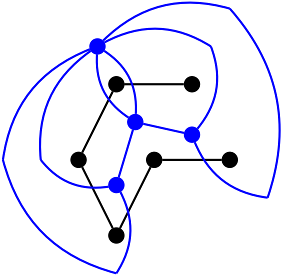

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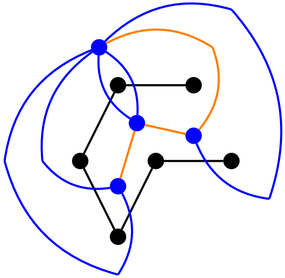

The spanning tree is acyclic.

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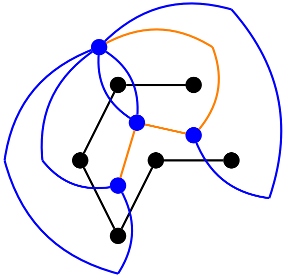

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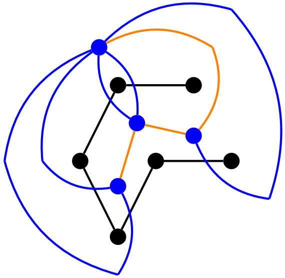

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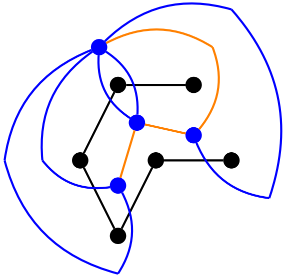

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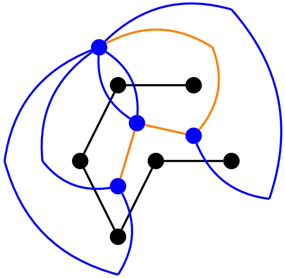

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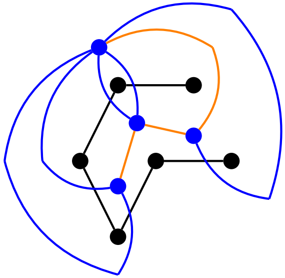

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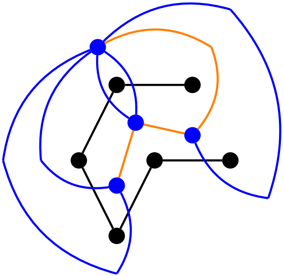

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The remaining dual edges form a spanning tree of G^*!

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Every spanning tree T in G has a dual spanning tree T' in G^* whose edges are edges in G^* which are not dual to T.

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This is called the method of "interdigitating spanning trees".

Summary

- Trees are minimally connected graphs (many equivalent definitions).
- We can perform induction on trees by removing a leaf.
- Planar graphs can be drawn without edge crossings.
- Trees are planar graphs.
- Each planar graph G has a dual planar graph G^* where the faces of G become the vertices of G^*.
- A cycle in G is a cut in G^* and vice versa.
- Each spanning tree in G has a dual spanning tree in G^*.
- This proves Euler's Formula: $v + f = e + 2$.