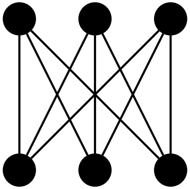
A Graph Puzzle



Can you draw this graph so that no edges cross?

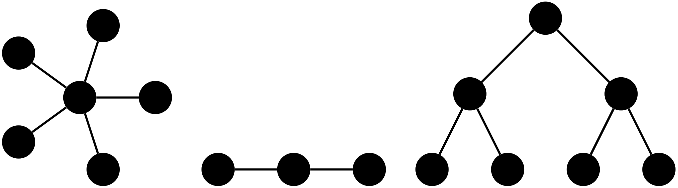
Think of the top nodes as “houses” and the bottom nodes as “utilities” (electricity, water, gas).

Can we have non-overlapping pipes?

Today: We study special graphs (trees and planar graphs).

Trees

A **tree** is a connected acyclic graph.



Trees are *minimally connected* graphs.

Equivalent definitions:

- Connected and acyclic.
- Connected and has $|V| - 1$ edges.
- Connected and the removal of *any* edge disconnects it.
- Acyclic and any *new* edge creates a cycle.

Removing a Leaf

A **leaf** is a vertex of degree one.

Lemma: After removing a leaf from a connected graph, the graph remains connected.

Proof.

- Let x be the leaf.
- For any vertices $u, v \in T$ which are not x, there is a path from u to v (T is connected). This path does not use x.
- After removing x, the path still exists, so the graph is still connected. $\square$

We need to remove leaves to do induction.

Equivalence of Tree Definitions

T is connected and acyclic $\iff$ T is connected and has $|V| - 1$ edges.

Proof ($\implies$).

- Use induction on $|V|$. The base case is easy.
- Consider a tree with $|V| \geq 2$.
- Take a path until you get stuck. Since there are no cycles, you must get stuck at a leaf. Remove the leaf.
- Resulting tree has $|V| - 1$ vertices and $|V| - 2$ edges (by induction). So our tree had $|V| - 1$ edges.

Equivalence of Tree Definitions

T is connected and acyclic $\iff$ T is connected and has $|V|-1$ edges.

Proof ($\Longleftarrow$).

- Handshaking Lemma: $\sum_{v\in V} \deg v = 2|E| = 2|V| - 2$.
- Since T is connected, $\deg v \geq 1$ for all $v \in V$.
- If $\deg v \geq 2$ for all $v \in V$, then $\sum_{v\in V} \deg v \geq 2|V|$ which is impossible.
- Thus there are at least two leaves.
- Removing the leaf, the resulting graph is connected and acyclic (by induction).
- Thus our tree was also connected and acyclic (leaves cannot be in cycles). $\square$

Trees Have Unique Paths

Lemma: In a tree, for any $x, y \in V$, there exists a *unique* path from x to y.

Proof.

- There exists a path: trees are connected.
- If there are two different paths from x to y, then take the first path from x to y and the second path back to x.
- This yields a cycle.
- Trees have no cycles, so the path is unique. $\square$

Equivalence of Tree Definitions

T is connected and acyclic $\iff$ T is connected and the removal of any edge disconnects T.

Proof ($\implies$).

- Consider removing $\{x, y\} \in E$.
- The edge $\{x, y\}$ must be the unique path from x to y in T.
- So removing $\{x, y\}$ disconnects x and y. $\square$

Proof ($\impliedby$).

- Remove the edge $\{x, y\}$.
- Now T is disconnected, so $\{x, y\}$ could not have been part of a cycle.
- T has no cycles. $\square$

Equivalence of Tree Definitions

T is connected and acyclic $\iff$ T is acyclic and any *new* edge creates a cycle.

Proof ($\Longrightarrow$).

- Add the new edge $\{x, y\}$.
- There was already a unique path from x to y, so this produces a cycle. $\square$

Proof ($\Longleftarrow$).

- For $x, y \in T$, if $\{x, y\} \in E$, then x and y are connected.
- Otherwise, adding $\{x, y\}$ creates a cycle, so there must have already been a path from x to y.
- Thus T is connected. $\square$

Planar Graphs

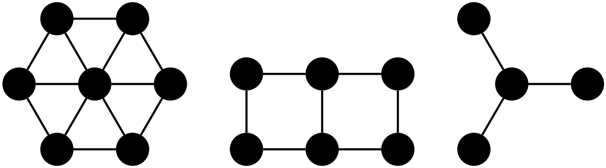
A **planar graph** is a graph which can be drawn in the plane with no edge crossings.

Theorem: Trees are planar.

Proof.

- Use induction on the number of vertices.
- For a tree with one vertex, this is easy.
- Consider a tree with $|V| \geq 2$ vertices.
- Remove a leaf u (connected to v). The resulting tree has fewer vertices, so it can be drawn without edge crossings.
- Zoom in on v. Since v has finitely many attached edges, there must be room to draw $\{u, v\}$ without crossings. $\square$

Euler's Formula



A **face** is a connected region of the plane.

This includes the infinite face!

How many vertices, faces, edges for the examples above?

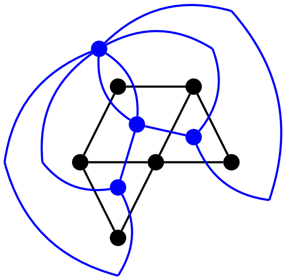
- $v = 7$, $f = 7$, $e = 12$.
- $v = 6$, $f = 3$, $e = 7$.
- $v = 4$, $f = 1$, $e = 3$.

Euler's Formula: $v + f = e + 2$.

Trees: $|V| + 1 = |V| - 1 + 2$.

Planar Duality

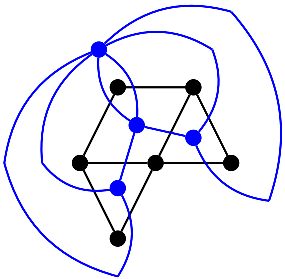
Given a connected planar graph G, we define the **dual planar graph** G^*:



- Each **face** in G becomes a **vertex** in G^*.
- Each **edge** in G corresponds to an **edge** in G^*.
- Technically, we should say *a* dual, instead of *the* dual—there may be multiple planar duals for G.

The Dual of the Dual

If G^* is a planar dual of G, then G is a planar dual of G^*.

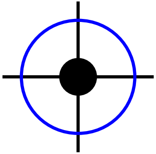


Just stare at it!

Proof of the Dual of the Dual

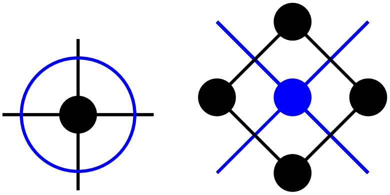
Proof.

- Look at a vertex.

  In G^*, the lines going through the edges incident to the vertex define a face.
- The face encloses the vertex.
- Thus, the vertices of G correspond to the faces of G^*.
- We already know that the edges of G and G^* correspond to each other. $\square$

Cycle-Cut Duality



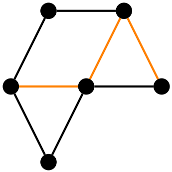
The previous argument is a special case of cycle-cut duality.

A **cut** is a set of edges which, if removed, separates a set of vertices from the rest of the vertices.

Cycle-Cut Duality: A cycle in G corresponds to a cut in G^*.

- A cycle encloses some faces of G. These faces correspond to vertices in G^*.
- The dual edges to the cycle form a cut.
- Since G is a dual of G^*, then a simple cut in G corresponds to a cycle in G^*.

Cuts & Connectedness



Consider a set of edges with no cuts. The remaining edges must form a connected graph.

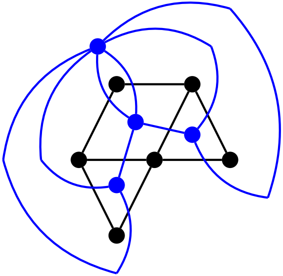
- Equivalently: If the remaining edges do not connect their vertices, then there must be a cut separating the vertices.

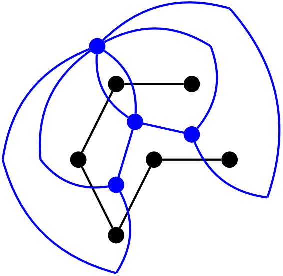
If a set of edges connects their vertices, then the remaining edges must not have any cuts for these vertices.

Spanning Trees

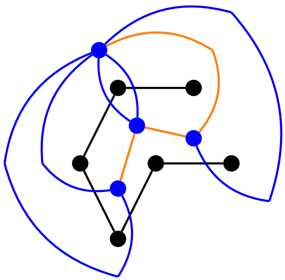
Start with a connected planar graph G. Find a **spanning tree**: a set of edges which form a tree in G.

- *Spanning* means that all vertices should be used.





Dual Spanning Trees



The spanning tree is acyclic.

- By cycle-cut duality, the dual edges have no cuts.
- So, the remaining dual edges are connected.

The spanning tree is connected.

- So, the remaining edges (not shown) have no cuts.
- So the remaining dual edges are acyclic.

The remaining dual edges form a spanning tree of G^*!

Spanning Trees to Euler's Formula

Every spanning tree T in G has a dual spanning tree T' in G^* whose edges are edges in G^* which are not dual to T.

Let e_T be the number of edges in T and $e_{T'}$ be the number of edges in T'.

So, $e = e_T + e_{T'}$.

- $e_T = v - 1$.
- $e_{T'} = f - 1$.

So $v + f = e + 2$. Euler's Formula!!!

This is called the method of "interdigitating spanning trees".

Summary

- Trees are minimally connected graphs (many equivalent definitions).
- We can perform induction on trees by removing a leaf.
- Planar graphs can be drawn without edge crossings.
- Trees are planar graphs.
- Each planar graph G has a dual planar graph G^* where the faces of G become the vertices of G^*.
- A cycle in G is a cut in G^* and vice versa.
- Each spanning tree in G has a dual spanning tree in G^*.
- This proves Euler's Formula: $v + f = e + 2$.