Stable Marriage

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The men and women have preferences:

Men	Preferences
A	$5 > 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$B > A > D > E > C$
4	$B > C > E > A > D$
5	$D > B > E > A > C$

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How should we match the men and women?

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Rogue Couples

Suppose that we pair together $(A, 1)$ and $(B, 2)$.

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The matching is called **stable** when there are no rogue couples.

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Question for today: How can we find a stable matching?

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This is called a **rogue couple**.

A **matching** is when we pair each man with a unique woman.

The matching is called **stable** when there are no rogue couples.

Question for today: How can we find a stable matching? Do stable matchings even exist?

Stable Matching Example

Men	Preferences
A	$5 > 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$B > A > D > E > C$
4	$B > C > E > A > D$
5	$D > B > E > A > C$

Is the matching $(A, 1)$, $(B, 2)$, $(C, 3)$, $(D, 4)$, $(E, 5)$ stable?

Stable Matching Example

Men	Preferences
A	$5 > 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$B > A > D > E > C$
4	$B > C > E > A > D$
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Is the matching $(A, 1)$, $(B, 2)$, $(C, 3)$, $(D, 4)$, $(E, 5)$ stable?

- Do you spot any rogue couples?

Stable Matching Example

Men	Preferences
A	$5 > 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$B > A > D > E > C$
4	$B > C > E > A > D$
5	$D > B > E > A > C$

Is the matching $(A, 1)$, $(B, 2)$, $(C, 3)$, $(D, 4)$, $(E, 5)$ stable?

- Do you spot any rogue couples?
- One example of a rogue couple: $(D, 5)$.

2×2 Stable Matchings

Men preferences match women preferences perfectly:

Men	Preferences
A	$1 > 2$
B	$2 > 1$

Women	Preferences
1	$A > B$
2	$B > A$

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There is a unique stable matching: $(A, 1)$, $(B, 2)$.

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Men have the same preferences:

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$(A,1)$ and $(B,2)$ is stable, since A and 1 are happy.

- Observation: If a man and woman both like each other best, they must be together in any stable matching.

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Men and women preferences clash:

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A	$1 > 2$
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Women	Preferences
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Here, there are two stable pairings:

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- $(A,1)$, $(B,2)$.

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Here, there are two stable pairings:

- $(A, 1)$, $(B, 2)$. Stable because the men are happy.

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- $(A,1)$, $(B,2)$. Stable because the men are happy.
- $(A,2)$, $(B,1)$.

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So, there may be multiple stable matchings. But, so far we have always been able to find at least one stable matching.

Stable Roommates

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- $(A,B),(C,D)$: (B,C) is a rogue pair.

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Gale-Shapley Algorithm

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- On each "day", each man without a partner proposes to the women highest up in his list.

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- At the end of each "day", each woman *tentatively* accepts her most preferred suitor and rejects every other man.

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Since then, the algorithm has found many applications:

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Here is the algorithm (men propose version).

- On each "day", each man without a partner proposes to the women highest up in his list.
- At the end of each "day", each woman *tentatively* accepts her most preferred suitor and rejects every other man.
- Terminate when every woman has a suitor.

Algorithm Example, Day 1

Men	Preferences
A	$5 > 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$B > A > D > E > C$
4	$B > C > E > A > D$
5	$D > B > E > A > C$

Algorithm Example, Day 1

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Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$B > A > D > E > C$
4	$B > C > E > A > D$
5	$D > B > E > A > C$

Men propose:

Algorithm Example, Day 1

Men	Preferences		Women	Preferences
A	$5 > 4 > 3 > 1 > 2$		1	$B > D > C > A > E$
B	$3 > 4 > 1 > 5 > 2$		2	$B > D > A > E > C$
C	$5 > 4 > 1 > 3 > 2$		3	$B > A > D > E > C$
D	$5 > 2 > 3 > 1 > 4$		4	$B > C > E > A > D$
E	$5 > 1 > 3 > 2 > 4$		5	$D > B > E > A > C$

Men propose:

- B proposes to 3.

Algorithm Example, Day 1

Men	Preferences
A	$5 > 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$B > A > D > E > C$
4	$B > C > E > A > D$
5	$D > B > E > A > C$

Men propose:

- B proposes to 3.
- A, C, D, and E propose to 5.

Algorithm Example, Day 1

Men	Preferences		Women	Preferences
A	$5 > 4 > 3 > 1 > 2$		1	$B > D > C > A > E$
B	$3 > 4 > 1 > 5 > 2$		2	$B > D > A > E > C$
C	$5 > 4 > 1 > 3 > 2$		3	$B > A > D > E > C$
D	$5 > 2 > 3 > 1 > 4$		4	$B > C > E > A > D$
E	$5 > 1 > 3 > 2 > 4$		5	$D > B > E > A > C$

Men propose:

- B proposes to 3.
- A, C, D, and E propose to 5.

Women respond:

Algorithm Example, Day 1

Men	Preferences
A	$5 > 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$B > A > D > E > C$
4	$B > C > E > A > D$
5	$D > B > E > A > C$

Men propose:

- B proposes to 3.
- A, C, D, and E propose to 5.

Women respond:

- 3 tentatively accepts B.

Algorithm Example, Day 1

Men	Preferences
A	$5 > 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$B > A > D > E > C$
4	$B > C > E > A > D$
5	$D > B > E > A > C$

Men propose:

- *B* proposes to 3.
- *A*, *C*, *D*, and *E* propose to 5.

Women respond:

- 3 tentatively accepts *B*.
- 5 tentatively accepts *D*; rejects *A*, *C*, *E*.

Algorithm Example, Day 2

Men	Preferences
A	~~5~~ $> 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	~~5~~ $> 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	~~5~~ $> 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	**B** $> A > D > E > C$
4	$B > C > E > A > D$
5	**D** $> B > E > A > C$

Algorithm Example, Day 2

Men	Preferences
A	~~5~~ $> 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	~~5~~ $> 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	~~5~~ $> 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > C > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

Algorithm Example, Day 2

Men	Preferences
A	~~5~~ $> 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	~~5~~ $> 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	~~5~~ $> 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > C > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

- E proposes to 1.

Algorithm Example, Day 2

Men	Preferences
A	$\not 5 > 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$\not 5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$\not 5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > C > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

- *E* proposes to 1.
- *A* and *C* propose to 4.

Algorithm Example, Day 2

Men	Preferences
A	~~5~~ $> 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	~~5~~ $> 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	~~5~~ $> 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > C > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

- E proposes to 1.
- A and C propose to 4.

Women respond:

Algorithm Example, Day 2

Men	Preferences
A	~~5~~ $> 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	~~5~~ $> 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	~~5~~ $> 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	**B** $> A > D > E > C$
4	$B > C > E > A > D$
5	**D** $> B > E > A > C$

Men propose:

- *E* proposes to 1.
- *A* and *C* propose to 4.

Women respond:

- 1 tentatively accepts *E*.

Algorithm Example, Day 2

Men	Preferences
A	~~5~~ $> 4 > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	~~5~~ $> 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	~~5~~ $> 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > E$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > C > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

- E proposes to 1.
- A and C propose to 4.

Women respond:

- 1 tentatively accepts E.
- 4 tentatively accepts C; rejects A.

Algorithm Example, Day 3

Men	Preferences
A	~~5~~ > ~~4~~ > 3 > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > 1 > 3 > 2 > 4

Women	Preferences
1	*B* > *D* > *C* > *A* > **E**
2	*B* > *D* > *A* > *E* > *C*
3	**B** > *A* > *D* > *E* > *C*
4	*B* > **C** > *E* > *A* > *D*
5	**D** > *B* > *E* > *A* > *C*

Algorithm Example, Day 3

Men	Preferences
A	$\not{5} > \not{4} > 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$\not{5} > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$\not{5} > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > \mathbf{E}$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > \mathbf{C} > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

Algorithm Example, Day 3

Men	Preferences
A	~~5~~ > ~~4~~ > 3 > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > 1 > 3 > 2 > 4

Women	Preferences
1	*B* > *D* > *C* > *A* > **E**
2	*B* > *D* > *A* > *E* > *C*
3	**B** > *A* > *D* > *E* > *C*
4	*B* > **C** > *E* > *A* > *D*
5	**D** > *B* > *E* > *A* > *C*

Men propose:

- *A* proposes to 3.

Algorithm Example, Day 3

Men	Preferences
A	~~5~~ $>$ ~~4~~ $> 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	~~5~~ $> 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	~~5~~ $> 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > \mathbf{E}$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > \mathbf{C} > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

- *A* proposes to 3.

Women respond:

Algorithm Example, Day 3

Men	Preferences
A	~~5~~ > ~~4~~ > 3 > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > 1 > 3 > 2 > 4

Women	Preferences
1	*B* > *D* > *C* > *A* > **E**
2	*B* > *D* > *A* > *E* > *C*
3	**B** > *A* > *D* > *E* > *C*
4	*B* > **C** > *E* > *A* > *D*
5	**D** > *B* > *E* > *A* > *C*

Men propose:

- *A* proposes to 3.

Women respond:

- 3 rejects *A* in favor of *B*.

Algorithm Example, Day 4

Men	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > 1 > 3 > 2 > 4

Women	Preferences
1	*B* > *D* > *C* > *A* > **E**
2	*B* > *D* > *A* > *E* > *C*
3	**B** > *A* > *D* > *E* > *C*
4	*B* > **C** > *E* > *A* > *D*
5	**D** > *B* > *E* > *A* > *C*

Algorithm Example, Day 4

Men	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > 1 > 3 > 2 > 4

Women	Preferences
1	*B* > *D* > *C* > *A* > **E**
2	*B* > *D* > *A* > *E* > *C*
3	**B** > *A* > *D* > *E* > *C*
4	*B* > **C** > *E* > *A* > *D*
5	**D** > *B* > *E* > *A* > *C*

Men propose:

Algorithm Example, Day 4

Men	Preferences
A	$\not 5 > \not 4 > \not 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$\not 5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$\not 5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > \mathbf{E}$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > \mathbf{C} > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

- *A* proposes to 1.

Algorithm Example, Day 4

Men	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > 1 > 3 > 2 > 4

Women	Preferences
1	*B* > *D* > *C* > *A* > **E**
2	*B* > *D* > *A* > *E* > *C*
3	**B** > *A* > *D* > *E* > *C*
4	*B* > **C** > *E* > *A* > *D*
5	**D** > *B* > *E* > *A* > *C*

Men propose:

- *A* proposes to 1.

Women respond:

Algorithm Example, Day 4

Men	Preferences
A	$\not 5 > \not 4 > \not 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$\not 5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$\not 5 > 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > A > \mathbf{E}$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > \mathbf{C} > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

- *A* proposes to 1.

Women respond:

- 1 rejects *E* in favor of *A*.

Algorithm Example, Day 5

Men	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > ~~1~~ > 3 > 2 > 4

Women	Preferences
1	*B* > *D* > *C* > **A** > *E*
2	*B* > *D* > *A* > *E* > *C*
3	**B** > *A* > *D* > *E* > *C*
4	*B* > **C** > *E* > *A* > *D*
5	**D** > *B* > *E* > *A* > *C*

Algorithm Example, Day 5

Men	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > ~~1~~ > 3 > 2 > 4

Women	Preferences
1	$B > D > C >$ **A** $> E$
2	$B > D > A > E > C$
3	**B** $> A > D > E > C$
4	$B >$ **C** $> E > A > D$
5	**D** $> B > E > A > C$

Men propose:

Algorithm Example, Day 5

Men	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > ~~1~~ > 3 > 2 > 4

Women	Preferences
1	*B* > *D* > *C* > **A** > *E*
2	*B* > *D* > *A* > *E* > *C*
3	**B** > *A* > *D* > *E* > *C*
4	*B* > **C** > *E* > *A* > *D*
5	**D** > *B* > *E* > *A* > *C*

Men propose:

- *E* proposes to 3.

Algorithm Example, Day 5

Men	Preferences
A	$\not 5 > \not 4 > \not 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$\not 5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$\not 5 > \not 1 > 3 > 2 > 4$

Women	Preferences
1	$B > D > C > \mathbf{A} > E$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > \mathbf{C} > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

- E proposes to 3.

Women respond:

Algorithm Example, Day 5

Men	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > ~~1~~ > 3 > 2 > 4

Women	Preferences
1	$B > D > C >$ **A** $> E$
2	$B > D > A > E > C$
3	**B** $> A > D > E > C$
4	$B >$ **C** $> E > A > D$
5	**D** $> B > E > A > C$

Men propose:

- *E* proposes to 3.

Women respond:

- 3 rejects *E* in favor of *B*.

Algorithm Example, Day 6

Men	Preferences	Women	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2	1	*B* > *D* > *C* > **A** > *E*
B	3 > 4 > 1 > 5 > 2	2	*B* > *D* > *A* > *E* > *C*
C	~~5~~ > 4 > 1 > 3 > 2	3	**B** > *A* > *D* > *E* > *C*
D	5 > 2 > 3 > 1 > 4	4	*B* > **C** > *E* > *A* > *D*
E	~~5~~ > ~~1~~ > ~~3~~ > 2 > 4	5	**D** > *B* > *E* > *A* > *C*

Algorithm Example, Day 6

Men	Preferences
A	$\not 5 > \not 4 > \not 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$\not 5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$\not 5 > \not 1 > \not 3 > 2 > 4$

Women	Preferences
1	$B > D > C > \mathbf{A} > E$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > \mathbf{C} > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

Algorithm Example, Day 6

Men	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > ~~1~~ > ~~3~~ > 2 > 4

Women	Preferences
1	*B* > *D* > *C* > **A** > *E*
2	*B* > *D* > *A* > *E* > *C*
3	**B** > *A* > *D* > *E* > *C*
4	*B* > **C** > *E* > *A* > *D*
5	**D** > *B* > *E* > *A* > *C*

Men propose:

- *E* proposes to 2.

Algorithm Example, Day 6

Men	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > ~~1~~ > ~~3~~ > 2 > 4

Women	Preferences
1	$B > D > C >$ **A** $> E$
2	$B > D > A > E > C$
3	**B** $> A > D > E > C$
4	$B >$ **C** $> E > A > D$
5	**D** $> B > E > A > C$

Men propose:

- E proposes to 2.

Women respond:

Algorithm Example, Day 6

Men	Preferences
A	$\not 5 > \not 4 > \not 3 > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$\not 5 > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$\not 5 > \not 1 > \not 3 > 2 > 4$

Women	Preferences
1	$B > D > C > \mathbf{A} > E$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > \mathbf{C} > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

- E proposes to 2.

Women respond:

- 2 tentatively accepts E.

Algorithm Example, Day 6

Men	Preferences
A	~~5~~ > ~~4~~ > ~~3~~ > 1 > 2
B	3 > 4 > 1 > 5 > 2
C	~~5~~ > 4 > 1 > 3 > 2
D	5 > 2 > 3 > 1 > 4
E	~~5~~ > ~~1~~ > ~~3~~ > 2 > 4

Women	Preferences
1	$B > D > C >$ **A** $> E$
2	$B > D > A > E > C$
3	**B** $> A > D > E > C$
4	$B >$ **C** $> E > A > D$
5	**D** $> B > E > A > C$

Men propose:

- E proposes to 2.

Women respond:

- 2 tentatively accepts E.

Since all women now have a suitor, the algorithm terminates with the pairing $(A, 1)$, $(B, 3)$, $(C, 4)$, $(D, 5)$, $(E, 2)$.

Algorithm Termination

Theorem: The Gale-Shapley algorithm terminates in finite time for any stable marriage instance.

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- So, each day that the algorithm runs, at least one man gets rejected.

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- So, each day that the algorithm runs, at least one man crosses off one woman from his preference list.

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- If no man gets rejected, the algorithm immediately terminates.
- So, each day that the algorithm runs, at least one man gets rejected.
- So, each day that the algorithm runs, at least one man crosses off one woman from his preference list.
- If there are n men, the men's preference lists have a total of n^2 entries, so the algorithm terminates in $\leq n^2$ days. □

Algorithm Termination

If every man has a different first choice, how many days does the algorithm require?

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On the other hand, there *are* instances which require $\Omega(n^2)$ days to complete, where n is the number of men/women.

There can be *exponentially many* stable marriages.

Algorithm Correctness

Theorem: The Gale-Shapley algorithm output is stable.

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- Suppose the result of the algorithm is not stable, i.e., there is a rogue couple (M, W).
- Say that the algorithm pairs (M, W') and (W, M').

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- Suppose the result of the algorithm is not stable, i.e., there is a rogue couple (M, W).
- Say that the algorithm pairs (M, W') and (W, M').
- So, M prefers W over W', and W prefers M over M'.

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- So, M proposes to W before he proposes to W'.

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- Suppose the result of the algorithm is not stable, i.e., there is a rogue couple (M, W).
- Say that the algorithm pairs (M, W') and (W, M').
- So, M prefers W over W', and W prefers M over M'.
- So, M proposes to W before he proposes to W'.
- But, since M proposes to W' later, that means he must have gotten rejected by W.

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- That means W must have found a better guy than M.

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- So, M proposes to W before he proposes to W'.
- But, since M proposes to W' later, that means he must have gotten rejected by W.
- That means W must have found a better guy than M. But W's suitors can only get better each day!

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- This means W must end up with a guy she likes better than M.

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- So, M prefers W over W', and W prefers M over M'.
- So, M proposes to W before he proposes to W'.
- But, since M proposes to W' later, that means he must have gotten rejected by W.
- That means W must have found a better guy than M. But W's suitors can only get better each day!
- This means W must end up with a guy she likes better than M. Contradiction. □

Which Stable Matching?

We saw that there can be multiple stable matchings.

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Recall: The stable matching $(A, 1)$, $(B, 2)$ favors the guys, and $(A, 2)$, $(B, 1)$ favors the gals.

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Run Gale-Shapley.

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Run Gale-Shapley. The algorithm ends in one day, and the pairing is $(A, 1)$, $(B, 2)$.

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Run Gale-Shapley. The algorithm ends in one day, and the pairing is $(A, 1)$, $(B, 2)$.

Does the Gale-Shapley algorithm favor the men?

Optimality

We say that a stable matching is **optimal** for a man if his partner in this matching is the best possible partner he can have, *out of all possible stable matchings*.

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Which stable matchings are optimal for A?

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Women	Preferences
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2	$A > B$

Which stable matchings are optimal for A?

- All possible stable matchings: $(A,1),(B,2)$ and $(A,2),(B,1)$.

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A	$1 > 2$
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Women	Preferences
1	$B > A$
2	$A > B$

Which stable matchings are optimal for A?

- All possible stable matchings: $(A,1),(B,2)$ and $(A,2),(B,1)$.
- What are all the possible women that A can end up with, *out of the possible stable matchings*?

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- Out of these women, which does A like best?

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- All possible stable matchings: $(A,1),(B,2)$ and $(A,2),(B,1)$.
- What are all the possible women that A can end up with, *out of the possible stable matchings*? $\{1,2\}$
- Out of these women, which does A like best? 1

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1	$B > A$
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Which stable matchings are optimal for A?

- All possible stable matchings: $(A,1),(B,2)$ and $(A,2),(B,1)$.
- What are all the possible women that A can end up with, *out of the possible stable matchings*? $\{1,2\}$
- Out of these women, which does A like best? 1
- Therefore, any stable matching in which A ends up with 1 is optimal for A.

Optimality vs. Most Preferred

Is a man's optimal partner the same as the first person in his preference list?

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2	$B > A$

What are the stable matchings?

Optimality vs. Most Preferred

Is a man's optimal partner the same as the first person in his preference list?

Men	Preferences		Women	Preferences
A	$1 > 2$		1	$B > A$
B	$1 > 2$		2	$B > A$

What are the stable matchings? Only $(A, 2), (B, 1)$.

Optimality vs. Most Preferred

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What are the stable matchings? Only $(A, 2), (B, 1)$.

In *any* stable matching, A must be partnered with 2. So the best he can do is 2 (his optimal partner).

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What are the stable matchings? Only $(A, 2), (B, 1)$.

In *any* stable matching, A must be partnered with 2. So the best he can do is 2 (his optimal partner).

So, A's optimal partner is *not* the same as his most preferred partner.

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What are the stable matchings? Only $(A, 2), (B, 1)$.

In *any* stable matching, A must be partnered with 2. So the best he can do is 2 (his optimal partner).

So, A's optimal partner is *not* the same as his most preferred partner. 1 is unattainable for A.

Male Optimality

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Here, $(A, 1), (B, 2)$ is male optimal because it is stable and every man gets his first choice.

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- A stable matching is **female pessimal** if every woman is with her pessimal partner.

Male Optimal Is Female Pessimal

Theorem: Consider two different stable matchings, μ_1 and μ_2. If every man likes μ_1 as much as μ_2, then every woman likes μ_2 as much as μ_1.

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- Every woman prefers any other stable matching over the male optimal matching.
- For a particular woman W, let her pessimal partner be M.
- There is some stable matching in which M and W are paired together.
- Since W prefers this matching over the male optimal matching, that means W must be matched with M in the male optimal matching.
- So, every woman is matched with her pessimal partner in the male optimal matching.

Summary

- We are given n men and n women with preference lists. We want a matching: an assignment of men to women.
- A rogue couple is a pair who prefer each other to their partners. A matching without a rogue couple is stable.
- The Gale-Shapley algorithm outputs a stable matching.
- A stable matching is male optimal if every man prefers this matching over any other stable matching.
- A stable matching is female pessimal if every woman prefers any other stable matching over this matching.
- The Gale-Shapley algorithm (with men proposing) is male optimal and female pessimal.