Stable Marriage

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¹Apologies for the heteronormativity.

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The men and women have preferences:

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Α	5 > 4 > 3 > 1 > 2	1	B > D > C > A > E
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	B > A > D > E > C
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How should we match the men and women?

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The matching is called **stable** when there are no rogue couples.

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Question for today: How can we find a stable matching?

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A matching is when we pair each man with a unique woman.

The matching is called **stable** when there are no rogue couples.

Question for today: How can we find a stable matching? Do stable matchings even exist?

Stable Matching Example

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Is the matching (A,1), (B,2), (C,3), (D,4), (E,5) stable?

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Do you spot any rogue couples?

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Is the matching (A, 1), (B, 2), (C, 3), (D, 4), (E, 5) stable?

- Do you spot any rogue couples?
- One example of a rogue couple: (D,5).

Men preferences match women preferences perfectly:

_	Men	Preferences	Women	Preferences
	Α	1 > 2	1	A > B
	В	2 > 1	2	B > A

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(A, 1) and (B, 2) is stable, since A and 1 are happy.

Observation: If a man and woman both like each other best, they must be together in any stable matching.

Men and women preferences clash:

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A	1 > 2	1	<i>B</i> > <i>A</i>
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So, there may be multiple stable matchings.

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So, there may be multiple stable matchings. But, so far we have always been able to find at least one stable matching.

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No matter how we assign roommates, we have a rogue pair.

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Gale-Shapley Algorithm

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- At the end of each "day", each woman *tentatively* accepts her most preferred suitor and rejects every other man.

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- Match organ transplant patients to organs.
- And more...

Here is the algorithm (men propose version).

- On each "day", each man without a partner proposes to the women highest up in his list.
- At the end of each "day", each woman *tentatively* accepts her most preferred suitor and rejects every other man.
- Terminate when every woman has a suitor.

Men	Preferences	Women	Preferences
A	5 > 4 > 3 > 1 > 2	1	B > D > C > A > E
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	B > A > D > E > C
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E	5 > 1 > 3 > 2 > 4	5	D > B > E > A > C

Men propose:

▶ *B* proposes to 3.

Men	Preferences	Women	Preferences
A	5 > 4 > 3 > 1 > 2	1	B > D > C > A > E
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	B > A > D > E > C
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Ε	5 > 1 > 3 > 2 > 4	5	D > B > E > A > C

- ▶ *B* proposes to 3.
- \blacktriangleright A, C, D, and E propose to 5.

Men	Preferences	Women	Preferences
A	5 > 4 > 3 > 1 > 2	1	B > D > C > A > E
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
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Men propose:

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Women respond:

Men	Preferences	Women	Preferences
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Men propose:

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- \blacktriangleright A, C, D, and E propose to 5.

Women respond:

► 3 tentatively accepts *B*.

Men	Preferences	Women	Preferences
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D	$5\!>\!2\!>\!3\!>\!1\!>\!4$	4	B > C > E > A > D
Ε	5 > 1 > 3 > 2 > 4	5	D > B > E > A > C

Men propose:

- B proposes to 3.
- \blacktriangleright A, C, D, and E propose to 5.

Women respond:

- ▶ 3 tentatively accepts *B*.
- ▶ 5 tentatively accepts *D*; rejects *A*, *C*, *E*.

Men	Preferences	Women	Preferences
A	5 > 4 > 3 > 1 > 2	1	B > D > C > A > E
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
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Е	5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > B > E > A > C$

Men propose:

► *E* proposes to 1.

Men	Preferences	Women	Preferences
A	5 > 4 > 3 > 1 > 2	1	B > D > C > A > E
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
D	5 > 2 > 3 > 1 > 4	4	B > C > E > A > D
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- E proposes to 1.
- ► A and C propose to 4.

Men	Preferences	Women	Preferences
A	5 > 4 > 3 > 1 > 2	1	B > D > C > A > E
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
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Men propose:

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Women respond:

▶ 1 tentatively accepts *E*.

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Men propose:

- E proposes to 1.
- A and C propose to 4.

Women respond:

- ▶ 1 tentatively accepts E.
- 4 tentatively accepts C; rejects A.

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В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
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Men propose:

-

► A proposes to 3.

Women respond:

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Men propose:

-

A proposes to 3.

Women respond:

► 3 rejects A in favor of B.

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В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
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D	$5\!>\!2\!>\!3\!>\!1\!>\!4$	4	$B > \mathbf{C} > E > A > D$
Ε	5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > B > E > A > C$

Men propose:

-

► A proposes to 1.

Women respond:

Men	Preferences	Women	Preferences
Α	5 > A > 3 > 1 > 2	1	$B > D > C > A > \mathbf{E}$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
D	$5\!>\!2\!>\!3\!>\!1\!>\!4$	4	$B > \mathbf{C} > E > A > D$
Ε	5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > B > E > A > C$

Men propose:

-

► A proposes to 1.

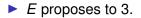
Women respond:

▶ 1 rejects *E* in favor of *A*.

Men	Preferences	Women	Preferences
A	5 > A > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
D	$5\!>\!2\!>\!3\!>\!1\!>\!4$	4	$B > \mathbf{C} > E > A > D$
E	5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > B > E > A > C$

Men	Preferences	Women	Preferences
A	5 > A > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
D	$5\!>\!2\!>\!3\!>\!1\!>\!4$	4	$B > \mathbf{C} > E > A > D$
Ε	<i>,</i> 5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > \mathbf{B} > \mathbf{E} > \mathbf{A} > \mathbf{C}$

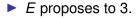
Men	Preferences	Women	Preferences
A	5 > A > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
D	$5\!>\!2\!>\!3\!>\!1\!>\!4$	4	$B > \mathbf{C} > E > A > D$
Е	<i>5</i> > 1⁄ > 3 > 2 > 4	5	$\mathbf{D} > B > E > A > C$



Men	Preferences	Women	Preferences
Α	5 > A > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
D	5 > 2 > 3 > 1 > 4	4	$B > \mathbf{C} > E > A > D$
Ε	5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > B > E > A > C$

Men propose:

-



Women respond:

Men	Preferences	Women	Preferences
Α	5 × 4 > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
D	$5\!>\!2\!>\!3\!>\!1\!>\!4$	4	$B > \mathbf{C} > E > A > D$
Ε	5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > B > E > A > C$

Men propose:

-

E proposes to 3.

Women respond:

► 3 rejects *E* in favor of *B*.

Men	Preferences	Women	Preferences
A	5 > A > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
D	$5\!>\!2\!>\!3\!>\!1\!>\!4$	4	$B > \mathbf{C} > E > A > D$
Е	5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > B > E > A > C$

Men	Preferences	Women	Preferences
A	5 > A > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
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E	5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > \mathbf{B} > \mathbf{E} > \mathbf{A} > \mathbf{C}$

Men propose:

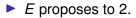
Men	Preferences	Women	Preferences
A	5 > A > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
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Ε	5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > B > E > A > C$

Men propose:

E proposes to 2.

Men	Preferences	Women	Preferences
A	5 > A > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
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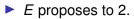
Men propose:



Women respond:

Men	Preferences	Women	Preferences
A	5 > A > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
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Men propose:

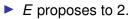


Women respond:

► 2 tentatively accepts *E*.

Men	Preferences	Women	Preferences
A	5 > A > 3 > 1 > 2	1	$B > D > C > \mathbf{A} > E$
В	3 > 4 > 1 > 5 > 2	2	B > D > A > E > C
С	5 > 4 > 1 > 3 > 2	3	$\mathbf{B} > A > D > E > C$
D	$5\!>\!2\!>\!3\!>\!1\!>\!4$	4	$B > \mathbf{C} > E > A > D$
E	5 > 1 > 3 > 2 > 4	5	$\mathbf{D} > B > E > A > C$

Men propose:



Women respond:

2 tentatively accepts E.

Since all women now have a suitor, the algorithm terminates with the pairing (A, 1), (B, 3), (C, 4), (D, 5), (E, 2).

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- So, each day that the algorithm runs, at least one man crosses off one woman from his preference list.
- If there are *n* men, the men's preference lists have a total of *n*² entries, so the algorithm terminates in ≤ *n*² days.

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There can be *exponentially many* stable marriages.

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- That means W must have found a better guy than M. But W's suitors can only get better each day!
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Recall: The stable matching (A, 1), (B, 2) favors the guys, and (A, 2), (B, 1) favors the gals.

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Does the Gale-Shapley algorithm favor the men?

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A	1 > 2		1	<i>B</i> > <i>A</i>
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Which stable matchings are optimal for A?

All possible stable matchings: (*A*, 1), (*B*, 2) and (*A*, 2), (*B*, 1).

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- All possible stable matchings: (A, 1), (B, 2) and (A, 2), (B, 1).
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- What are all the possible women that A can end up with, out of the possible stable matchings? {1,2}
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- Therefore, any stable matching in which A ends up with 1 is optimal for A.

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So, *A*'s optimal partner is *not* the same as his most preferred partner. 1 is unattainable for *A*.

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Here, (A, 1), (B, 2) is male optimal because it is stable and every man gets his first choice.

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- ► But W likes M' more than M, and M' likes W at least as much as his partner. This is a rogue couple.

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The Gale-Shapley algorithm is good for men.

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- A stable matching is **female pessimal** if every woman is with her pessimal partner.

Male Optimal Is Female Pessimal

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- Take a pair (M, W) in μ_1 which is not matched in μ_2 .
- If both *M* and *W* prefer μ₁, that means *M* and *W* like each other more than their partners in μ₂.
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- ▶ So, if *M* prefers μ_1 , then *W* prefers μ_2 . \Box

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► The male optimal matching is female pessimal.

More details:

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- There is some stable matching in which M and W are paired together.
- Since W prefers this matching over the male optimal matching, that means W must be matched with M in the male optimal matching.
- So, every woman is matched with her pessimal partner in the male optimal matching.

Summary

- We are given n men and n women with preference lists. We want a matching: an assignment of men to women.
- A rogue couple is a pair who prefer each other to their partners. A matching without a rogue couple is stable.
- ► The Gale-Shapley algorithm outputs a stable matching.
- A stable matching is male optimal if every man prefers this matching over any other stable matching.
- A stable matching is female pessimal if every woman prefers any other stable matching over this matching.
- The Gale-Shapley algorithm (with men proposing) is male optimal and female pessimal.