Stable Marriage

Suppose five men (A, B, C, D, E) and five women (1, 2, 3, 4, 5) enter a matchmaking service.[1]

The men and women have preferences:

Men	Preferences	Women	Preferences
A	$5 > 4 > 3 > 1 > 2$	1	$B > D > C > A > E$
B	$3 > 4 > 1 > 5 > 2$	2	$B > D > A > E > C$
C	$5 > 4 > 1 > 3 > 2$	3	$B > A > D > E > C$
D	$5 > 2 > 3 > 1 > 4$	4	$B > C > E > A > D$
E	$5 > 1 > 3 > 2 > 4$	5	$D > B > E > A > C$

How should we match the men and women?

[1] Apologies for the heteronormativity.

Rogue Couples

Suppose that we pair together $(A, 1)$ and $(B, 2)$.

- What happens if A likes 2 better than his current partner, and 2 likes A better than her current partner?
- There is an incentive to cheat, of course!

This is called a **rogue couple**.

A **matching** is when we pair each man with a unique woman.

The matching is called **stable** when there are no rogue couples.

Question for today: How can we find a stable matching? Do stable matchings even exist?

Stable Matching Example

Men	Preferences	Women	Preferences
A	$5 > 4 > 3 > 1 > 2$	1	$B > D > C > A > E$
B	$3 > 4 > 1 > 5 > 2$	2	$B > D > A > E > C$
C	$5 > 4 > 1 > 3 > 2$	3	$B > A > D > E > C$
D	$5 > 2 > 3 > 1 > 4$	4	$B > C > E > A > D$
E	$5 > 1 > 3 > 2 > 4$	5	$D > B > E > A > C$

Is the matching $(A, 1)$, $(B, 2)$, $(C, 3)$, $(D, 4)$, $(E, 5)$ stable?

- Do you spot any rogue couples?
- One example of a rogue couple: $(D, 5)$.

2×2 Stable Matchings

Men preferences match women preferences perfectly:

Men	Preferences	Women	Preferences
A	$1 > 2$	1	$A > B$
B	$2 > 1$	2	$B > A$

There is a unique stable matching: $(A, 1)$, $(B, 2)$.

Men have the same preferences:

Men	Preferences	Women	Preferences
A	$1 > 2$	1	$A > B$
B	$1 > 2$	2	$B > A$

$(A, 1)$ and $(B, 2)$ is stable, since A and 1 are happy.

- Observation: If a man and woman both like each other best, they must be together in any stable matching.

2×2 Stable Matchings

Men and women preferences clash:

Men	Preferences	Women	Preferences
A	$1 > 2$	1	$B > A$
B	$2 > 1$	2	$A > B$

Here, there are two stable pairings:

- $(A, 1)$, $(B, 2)$. Stable because the men are happy.
- $(A, 2)$, $(B, 1)$. Stable because the women are happy.

So, there may be multiple stable matchings. But, so far we have always been able to find at least one stable matching.

Stable Roommates

It is not obvious that stable matchings always exist.

Consider a variant: in the **stable roommates** problem we no longer have men/women.

Person	Preference
A	$B > C > D$
B	$C > A > D$
C	$A > B > D$
D	$A > B > C$

No matter how we assign roommates, we have a rogue pair.

- $(A, B), (C, D)$: (B, C) is a rogue pair.
- $(A, C), (B, D)$: (A, B) is a rogue pair.
- $(A, D), (B, C)$: (A, C) is a rogue pair.

Gale-Shapley Algorithm

Gale and Shapley gave an algorithm for finding a stable matching. Their work led to a Nobel Prize in Economics.

Since then, the algorithm has found many applications:

- Match new doctors to hospital residency programs.
- Match organ transplant patients to organs.
- And more...

Here is the algorithm (men propose version).

- On each "day", each man without a partner proposes to the women highest up in his list.
- At the end of each "day", each woman *tentatively* accepts her most preferred suitor and rejects every other man.
- Terminate when every woman has a suitor.

Algorithm Example, Day 1

Men	Preferences	Women	Preferences
A	$5 > 4 > 3 > 1 > 2$	1	$B > D > C > A > E$
B	$3 > 4 > 1 > 5 > 2$	2	$B > D > A > E > C$
C	$5 > 4 > 1 > 3 > 2$	3	$B > A > D > E > C$
D	$5 > 2 > 3 > 1 > 4$	4	$B > C > E > A > D$
E	$5 > 1 > 3 > 2 > 4$	5	$D > B > E > A > C$

Men propose:

- B proposes to 3.
- A, C, D, and E propose to 5.

Women respond:

- 3 tentatively accepts B.
- 5 tentatively accepts D; rejects A, C, E.

Algorithm Example, Day 2

Men	Preferences	Women	Preferences
A	$\cancel{5} > 4 > 3 > 1 > 2$	1	$B > D > C > A > E$
B	$3 > 4 > 1 > 5 > 2$	2	$B > D > A > E > C$
C	$\cancel{5} > 4 > 1 > 3 > 2$	3	$\mathbf{B} > A > D > E > C$
D	$5 > 2 > 3 > 1 > 4$	4	$B > C > E > A > D$
E	$\cancel{5} > 1 > 3 > 2 > 4$	5	$\mathbf{D} > B > E > A > C$

Men propose:

- E proposes to 1.
- A and C propose to 4.

Women respond:

- 1 tentatively accepts E.
- 4 tentatively accepts C; rejects A.

Algorithm Example, Day 3

Men	Preferences	Women	Preferences
A	$\cancel{5} > \cancel{4} > 3 > 1 > 2$	1	$B > D > C > A > \mathbf{E}$
B	$3 > 4 > 1 > 5 > 2$	2	$B > D > A > E > C$
C	$\cancel{5} > 4 > 1 > 3 > 2$	3	$\mathbf{B} > A > D > E > C$
D	$5 > 2 > 3 > 1 > 4$	4	$B > \mathbf{C} > E > A > D$
E	$\cancel{5} > 1 > 3 > 2 > 4$	5	$\mathbf{D} > B > E > A > C$

Men propose:

- A proposes to 3.

Women respond:

- 3 rejects A in favor of B.

Algorithm Example, Day 4

Men	Preferences	Women	Preferences
A	$\cancel{5} > \cancel{4} > \cancel{3} > 1 > 2$	1	$B > D > C > A > \mathbf{E}$
B	$3 > 4 > 1 > 5 > 2$	2	$B > D > A > E > C$
C	$\cancel{5} > 4 > 1 > 3 > 2$	3	$\mathbf{B} > A > D > E > C$
D	$5 > 2 > 3 > 1 > 4$	4	$B > \mathbf{C} > E > A > D$
E	$\cancel{5} > 1 > 3 > 2 > 4$	5	$\mathbf{D} > B > E > A > C$

Men propose:

- A proposes to 1.

Women respond:

- 1 rejects E in favor of A.

Algorithm Example, Day 5

Men	Preferences	Women	Preferences
A	$\cancel{5} > \cancel{4} > \cancel{3} > 1 > 2$	1	$B > D > C > \mathbf{A} > E$
B	$3 > 4 > 1 > 5 > 2$	2	$B > D > A > E > C$
C	$\cancel{5} > 4 > 1 > 3 > 2$	3	$\mathbf{B} > A > D > E > C$
D	$5 > 2 > 3 > 1 > 4$	4	$B > \mathbf{C} > E > A > D$
E	$\cancel{5} > \cancel{1} > 3 > 2 > 4$	5	$\mathbf{D} > B > E > A > C$

Men propose:

- E proposes to 3.

Women respond:

- 3 rejects E in favor of B.

Algorithm Example, Day 6

Men	Preferences
A	$\cancel{5} > \cancel{4} > \cancel{3} > 1 > 2$
B	$3 > 4 > 1 > 5 > 2$
C	$\cancel{5} > 4 > 1 > 3 > 2$
D	$5 > 2 > 3 > 1 > 4$
E	$\cancel{5} > \cancel{1} > \cancel{3} > 2 > 4$

Women	Preferences
1	$B > D > C > \mathbf{A} > E$
2	$B > D > A > E > C$
3	$\mathbf{B} > A > D > E > C$
4	$B > \mathbf{C} > E > A > D$
5	$\mathbf{D} > B > E > A > C$

Men propose:

- E proposes to 2.

Women respond:

- 2 tentatively accepts E.

Since all women now have a suitor, the algorithm terminates with the pairing $(A,1)$, $(B,3)$, $(C,4)$, $(D,5)$, $(E,2)$.

Algorithm Termination

Theorem: The Gale-Shapley algorithm terminates in finite time for any stable marriage instance.

Proof.

- Each day of the algorithm, at least one man is proposing.
- If no man gets rejected, the algorithm immediately terminates.
- So, each day that the algorithm runs, at least one man gets rejected.
- So, each day that the algorithm runs, at least one man crosses off one woman from his preference list.
- If there are n men, the men's preference lists have a total of n^2 entries, so the algorithm terminates in $\leq n^2$ days. $\square$

Algorithm Termination

If every man has a different first choice, how many days does the algorithm require? Just one!

On the other hand, there *are* instances which require $\Omega(n^2)$ days to complete, where n is the number of men/women.

There can be *exponentially many* stable marriages.

Algorithm Correctness

Theorem: The Gale-Shapley algorithm output is stable.

Proof.

- Suppose the result of the algorithm is not stable, i.e., there is a rogue couple (M, W).
- Say that the algorithm pairs (M, W') and (W, M').
- So, M prefers W over W', and W prefers M over M'.
- So, M proposes to W before he proposes to W'.
- But, since M proposes to W' later, that means he must have gotten rejected by W.
- That means W must have found a better guy than M. But W's suitors can only get better each day!
- This means W must end up with a guy she likes better than M. Contradiction. $\square$

Which Stable Matching?

We saw that there can be multiple stable matchings. Which one does the Gale-Shapley algorithm output?

Men	Preferences
A	$1 > 2$
B	$2 > 1$

Women	Preferences
1	$B > A$
2	$A > B$

Recall: The stable matching $(A,1)$, $(B,2)$ favors the guys, and $(A,2)$, $(B,1)$ favors the gals.

Run Gale-Shapley. The algorithm ends in one day, and the pairing is $(A,1)$, $(B,2)$.

Does the Gale-Shapley algorithm favor the men?

Optimality

We say that a stable matching is **optimal** for a man if his partner in this matching is the best possible partner he can have, *out of all possible stable matchings*.

Men	Preferences
A	$1 > 2$
B	$2 > 1$

Women	Preferences
1	$B > A$
2	$A > B$

Which stable matchings are optimal for A?

- All possible stable matchings: $(A,1), (B,2)$ and $(A,2), (B,1)$.
- What are all the possible women that A can end up with, *out of the possible stable matchings*? $\{1,2\}$
- Out of these women, which does A like best? 1
- Therefore, any stable matching in which A ends up with 1 is optimal for A.

Optimality vs. Most Preferred

Is a man's optimal partner the same as the first person in his preference list?

Men	Preferences	Women	Preferences
A	$1 > 2$	1	$B > A$
B	$1 > 2$	2	$B > A$

What are the stable matchings? Only $(A, 2), (B, 1)$.

In *any* stable matching, A must be partnered with 2. So the best he can do is 2 (his optimal partner).

So, A's optimal partner is *not* the same as his most preferred partner. 1 is unattainable for A.

Male Optimality

We say that a stable matching is **male optimal** if *every* man is paired with his optimal partner.

- This is a pretty strong condition— all men are happy simultaneously!

Men	Preferences	Women	Preferences
A	$1 > 2$	1	$B > A$
B	$2 > 1$	2	$A > B$

Here, $(A, 1), (B, 2)$ is male optimal because it is stable and every man gets his first choice.

Male Optimality for Gale-Shapley

Theorem: The Gale-Shapley algorithm outputs a male optimal stable matching.

Proof.

- Suppose that the output is not male optimal.
- Consider the first day in which some man M is rejected by his optimal partner W (Well Ordering Principle).
- Since W rejects M, that means she had a man she likes better: call him M'.
- M' has not yet been rejected by his optimal woman. So, W is at least as good as the optimal woman for M'.
- Since W is optimal for M, there exists a stable matching in which M is paired with W.
- But W likes M' more than M, and M' likes W at least as much as his partner. This is a rogue couple. □

Notes on the Optimality Result

How do we modify the Gale-Shapley algorithm to favor the women?

- Switch the roles of men and women.
- In other words, the women propose.

The Gale-Shapley algorithm is good for men. Is it necessarily bad for women?

- A man is **pessimal** for a woman if, out of all stable matchings, this man is her least preferred partner.
- A stable matching is **female pessimal** if every woman is with her pessimal partner.

Male Optimal Is Female Pessimal

Theorem: Consider two different stable matchings, μ_1 and μ_2. If every man likes μ_1 as much as μ_2, then every woman likes μ_2 as much as μ_1.

Proof.

- Take a pair (M, W) in μ_1 which is not matched in μ_2.
- If both M and W prefer μ_1, that means M and W like each other more than their partners in μ_2.
- This is a rogue couple!
- So, if M prefers μ_1, then W prefers μ_2. □

Every man prefers the male optimal matching over any other stable matching. So, every woman prefers any other stable matching over the male optimal matching.

- The male optimal matching is female pessimal.

Male Optimal Is Female Pessimal

More details:

- Every woman prefers any other stable matching over the male optimal matching.
- For a particular woman W, let her pessimal partner be M.
- There is some stable matching in which M and W are paired together.
- Since W prefers this matching over the male optimal matching, that means W must be matched with M in the male optimal matching.
- So, every woman is matched with her pessimal partner in the male optimal matching.

Summary

- We are given n men and n women with preference lists. We want a matching: an assignment of men to women.
- A rogue couple is a pair who prefer each other to their partners. A matching without a rogue couple is stable.
- The Gale-Shapley algorithm outputs a stable matching.
- A stable matching is male optimal if every man prefers this matching over any other stable matching.
- A stable matching is female pessimal if every woman prefers any other stable matching over this matching.
- The Gale-Shapley algorithm (with men proposing) is male optimal and female pessimal.