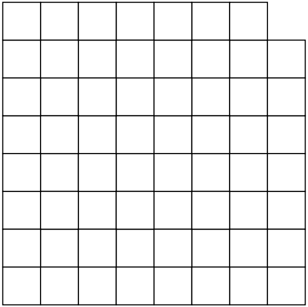
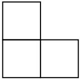
Domino Tilings



Can you tile the grid with L-shaped tiles?



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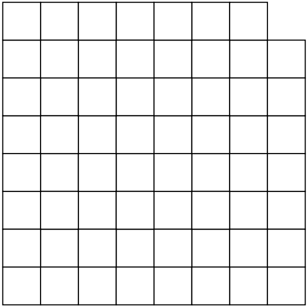

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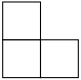

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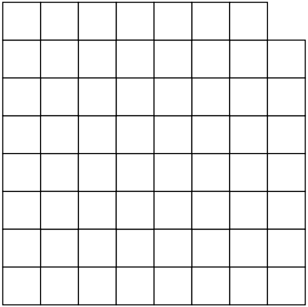

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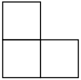

What about for a general $2^n \times 2^n$ grid and the hole is anywhere?

Gauss & Induction

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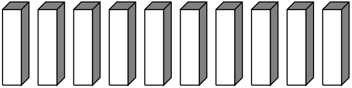
We will prove it too, using *induction*.

Knocking over Dominoes

Consider an infinite line of dominoes:



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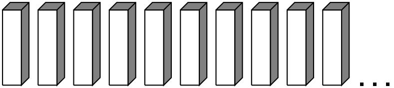

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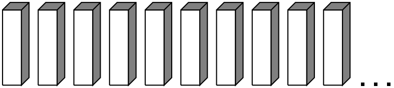

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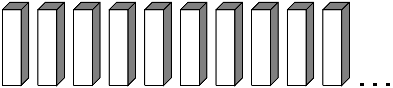

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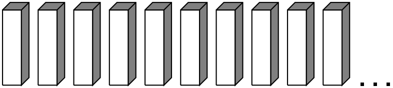

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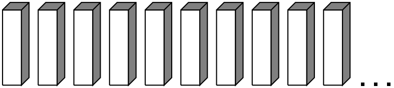

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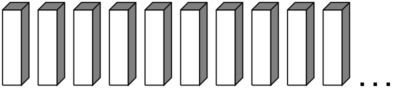

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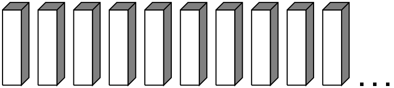

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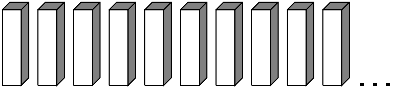

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This is the key idea behind induction.

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Principle of Induction: To prove a statement $\forall n \in \mathbb{N}\, P(n)$, it is enough to prove:

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Key idea: Proofs must be of finite length. The principle of induction lets us “cheat” and condense an infinitely long proof.

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This completes the proof. $\square$

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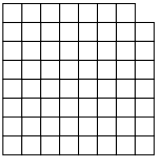
Domino Tiling

For a positive integer n, consider the $2^n \times 2^n$ grid with the upper-right corner missing.



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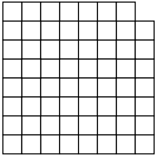

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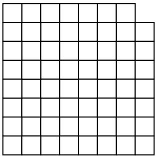

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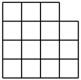
We are done!

Domino Tiling: Inductive Step

Now let us try $n = 2$.



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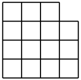

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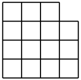

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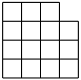

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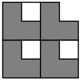

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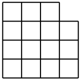

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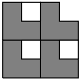

We failed!

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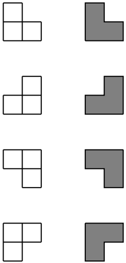

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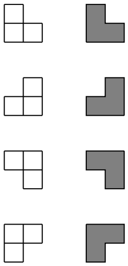

The missing hole can be anywhere, but we can rotate our L-tile to accommodate all cases.

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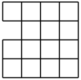

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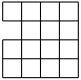

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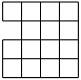

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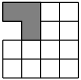

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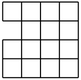

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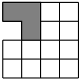

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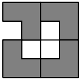

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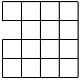

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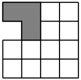

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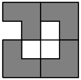

- Can you complete the proof? $\square$

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If your inductive claim does *not* contain enough information, reformulate your theorem to include this necessary information.

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- We cannot make change for 11 cents.

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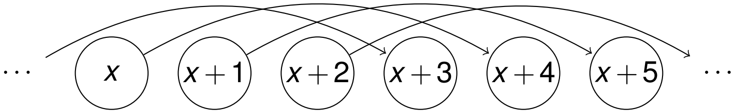
If induction is climbing a ladder one step at a time... here we can climb the ladder four steps at a time.

Visualizing Change

Stare at this graph.



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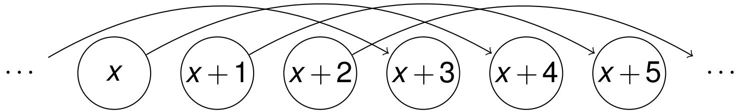
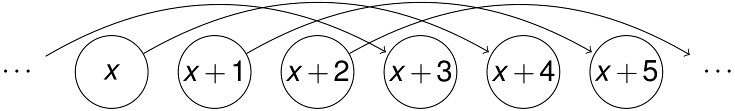


We can think of this as four separate ladders:

- $P(0) \implies P(4)$, $P(4) \implies P(8)$, $P(8) \implies P(12), \ldots$
- $P(1) \implies P(5)$, $P(5) \implies P(9)$, $P(9) \implies P(13), \ldots$
- $P(2) \implies P(6)$, $P(6) \implies P(10)$, $P(10) \implies P(14), \ldots$
- $P(3) \implies P(7)$, $P(7) \implies P(11)$, $P(11) \implies P(15), \ldots$

Visualizing Change

Stare at this graph.



We can think of this as four separate ladders:

- $P(0) \implies P(4)$, $P(4) \implies P(8)$, $P(8) \implies P(12)$, ...
- $P(1) \implies P(5)$, $P(5) \implies P(9)$, $P(9) \implies P(13)$, ...
- $P(2) \implies P(6)$, $P(6) \implies P(10)$, $P(10) \implies P(14)$, ...
- $P(3) \implies P(7)$, $P(7) \implies P(11)$, $P(11) \implies P(15)$, ...

Idea: If we can make change for four consecutive numbers x, $x+1$, $x+2$, $x+3$, then we can make change for all $n \geq x$.

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- How do we make change for $x+4$? Make change for x, and then add a 4-cent coin. $\square$

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- Knock over dominoes, where *all previously knocked down dominoes* help knock over the next domino.

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Theorem: For any natural number $n \geq 2$, we can write n as a product of prime numbers.

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- Thus, $n+1$ is a product of primes. $\square$

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- Ordinary induction to prove $\forall n \in \mathbb{N}\ Q(n)$ is *equivalent* to using strong induction to prove $\forall n \in \mathbb{N}\ P(n)$.

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Strong induction is a different way to *apply* ordinary induction.

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- Since the first n and last n horses overlap, then all $n+1$ horses are the same color. ♠

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Spot the mistake!

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- The base case is usually easy so it is sometimes ignored.
- This costs you points on the midterm.

Summary

- To prove $\forall n \in \mathbb{N}\ P(n)$, prove:
  1. the base case $P(0)$, and
  2. for all $n \in \mathbb{N}$, assume $P(n)$ and prove $P(n+1)$.
- Domino tilings and moving the hole around:
  - Sometimes *strengthening* the claim makes it easier to prove!
- Strong induction: in the inductive step, assume $P(0), P(1), \ldots, P(n-1)$ in addition to $P(n)$.
- Strong induction is equivalent to ordinary induction.
- All horses are not the same color: you can make mistakes if you are not careful.