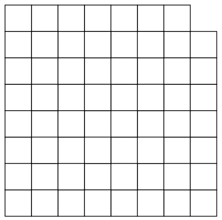
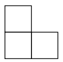
Domino Tilings



Can you tile the grid with L-shaped tiles?



What about for a general $2^n \times 2^n$ grid and the hole is anywhere?

Gauss & Induction

An old story: seven-year-old Gauss is in class. Teacher asks: what is $1+2+3+\cdots+100$?

- Gauss notices that the sum can be written as $(1+100)+(2+99)+(3+98)+\cdots+(50+51)$.
- 50 pairs of numbers, each pair sums to 101.
- The answer is 5050.

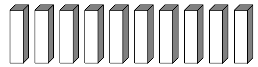
Gauss was proving the statement

$$\forall n \in \mathbb{N} \left(\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \right).$$

We will prove it too, using *induction*.

Knocking over Dominoes

Consider an infinite line of dominoes:

...

How do you knock them *all* down? Easy answer: Knock over the first one.

Why does domino 1 fall? You knocked it over.
Why does domino 2 fall? Domino 1 knocked it over.
⋮
Why does domino $n+1$ fall? Domino n knocked it over.

This is the key idea behind induction.

Principle of Mathematical Induction

Principle of Induction: To prove a statement $\forall n \in \mathbb{N}\ P(n)$, it is enough to prove:

1. $P(0)$;
2. $\forall n \in \mathbb{N}\ [P(n) \implies P(n+1)]$.

In symbols:

$$\forall n \in \mathbb{N}\ P(n) \equiv P(0) \wedge (\forall n \in \mathbb{N}\ [P(n) \implies P(n+1)]).$$

Why is induction helpful? We can *assume* that $P(n)$ is true, and then prove that $P(n+1)$ holds.

Step 1 is the **base case**. Step 2 is the **inductive step**. Assuming that $P(n)$ holds is called the **inductive hypothesis**.

More on Induction

Suppose we have proven:

1. $P(0)$;
2. $\forall n \in \mathbb{N}\ [P(n) \implies P(n+1)]$.

From Step 1, we have proven $P(0)$.

As a special case of Step 2, we have proven $P(0) \implies P(1)$. Since we know $P(0)$ holds, then we conclude that $P(1)$ holds.

As a special case of Step 2, we have proven $P(1) \implies P(2)$. Since we know $P(1)$ holds, then we conclude that $P(2)$ holds.

Understand the idea?

Key idea: Proofs must be of finite length. The principle of induction lets us "cheat" and condense an infinitely long proof.

Proving Gauss's Formula

For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} i = n(n+1)/2$.

- Base case: $P(0)$.
$$\sum_{i=0}^{0} i = \frac{0 \cdot 1}{2}.$$
The LHS and RHS are 0, so the base case holds.
- Inductive hypothesis: Assume $P(n)$, i.e., assume $\sum_{i=0}^{n} i = n(n+1)/2$ holds.
- **Important**: We assume $P(n)$ holds for *one unspecified* $n \in \mathbb{N}$. We do **NOT** assume $P(n)$ holds for *all n*.
- Inductive step: Prove $P(n+1)$.
$$\sum_{i=0}^{n+1} i = \sum_{i=0}^{n} i + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}.$$

This completes the proof. □

Better Triangle Inequality

Recall: For all $x, y \in \mathbb{R}$, $|x+y| \le |x| + |y|$ (Triangle Inequality).

Prove: For all *positive integers* n and real numbers $x_1, \ldots, x_n$, we have $|x_1 + \cdots + x_n| \le |x_1| + \cdots + |x_n|$.

- Statement: $P(n) = \forall x_1, \ldots, x_n \in \mathbb{R} \; |\sum_{i=1}^n x_i| \le \sum_{i=1}^n |x_i|$.
- Base case: Start with $P(1)$. $|x_1| \le |x_1|$ for all $x_1 \in \mathbb{R}$. Obviously true.
- Inductive hypothesis: For some $n \in \mathbb{N}$, assume that $|x_1 + \cdots + x_n| \le |x_1| + \cdots + |x_n|$ for all $x_1, \ldots, x_n \in \mathbb{R}$.
- Inductive step: Prove $\forall x_1, \ldots, x_{n+1} \in \mathbb{R} \; |\sum_{i=1}^{n+1} x_i| \le \sum_{i=1}^{n+1} |x_i|$. Let $x_1, \ldots, x_{n+1}$ be arbitrary real numbers.

$$\left|\sum_{i=1}^{n+1} x_i\right| = \left|\sum_{i=1}^{n} x_i + x_{n+1}\right| \le \left|\sum_{i=1}^{n} x_i\right| + |x_{n+1}| \le \sum_{i=1}^{n} |x_i| + |x_{n+1}|.$$

This proves $P(n+1)$. $\square$

Recursion & Induction

We *define objects* via **recursion**, and *prove statements* via **induction**.

- The two concepts are closely related.
- Let $a_0 := 1$, and for $n \in \mathbb{N}$, define $a_{n+1} := 2a_n$. (recursive definition)
- Prove: For all $n \in \mathbb{N}$, $a_n = 2^n$. How? (inductive proof)

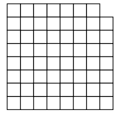
Recall from CS 61A: *tree recursion*.

- Example: Finding the height of a binary tree T.
- If T is a leaf, $\text{height}(T) = 1$.
- Otherwise, $\text{height}(T) = 1 + \max\{\text{height}(\text{left subtree}), \text{height}(\text{right subtree})\}$.

Just as we can do recursion on trees, we can prove facts about trees *inductively*. (Next topic: graph theory.)

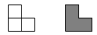
Domino Tiling

For a positive integer n, consider the $2^n \times 2^n$ grid with the upper-right corner missing.



Can we tile the grid with L-shaped tiles?

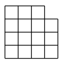
Base case, $n = 1$.



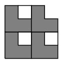
We are done!

Domino Tiling: Inductive Step

Now let us try $n = 2$.



Think of the 4×4 grid as four copies of the 2×2 grid. Apply inductive tiling?



We failed!

Strengthening the Inductive Hypothesis

Counterintuitive idea: Make the theorem *stronger*.

New Theorem: For any positive integer n, given a $2^n \times 2^n$ grid with any square missing, we can tile it with L-shaped tiles.

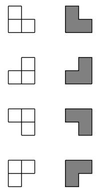
Counterintuitive?

- The theorem is now *harder* to prove, since the missing hole can be anywhere.
- However, in an inductive proof where we assume $P(n)$, we have more information at our disposal to prove $P(n+1)$.

Domino Tiling: Second Try

New Theorem: For any positive integer n, given a $2^n \times 2^n$ grid with any square missing, we can tile it with L-shaped tiles.

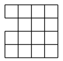
Now, there are four base cases.



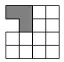
The missing hole can be anywhere, but we can rotate our L-tile to accommodate all cases.

Domino Tiling: Second Try

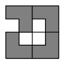
Again, try $n = 2$.



- Split 4×4 grid into four 2×2 grids.
- In the 2×2 grid with the missing square, tile with inductive hypothesis.



- Tile the other 2×2 grids with holes lining up using the (strengthened) inductive hypothesis.



- Can you complete the proof? $\square$

Strengthening the Inductive Hypothesis

Key idea: The inductive claim must contain information in order to propagate the claim from $P(n)$ to $P(n+1)$.

If your inductive claim does *not* contain enough information, reformulate your theorem to include this necessary information.

Making Change

You live in a country where there are only two types of coins: 4-cent coins and 5-cent coins.

Question: If I need x cents total, using only 4-cent and 5-cent coins, can you add up to exactly x cents?

- We cannot make change for amounts less than 4 cents.
- We cannot make change for 6 cents or 7 cents.
- We can make change for 8 cents with two 4-cent coins.
- We can make change for 9 cents with a 4-cent coin and a 5-cent coin.
- We can make change for 10 cents with two 5-cent coins.
- We cannot make change for 11 cents.

Think Inductively

Try to make change inductively.

If we can make change for x cents, we can make change for $x + 4$ cents (add a 4-cent coin).

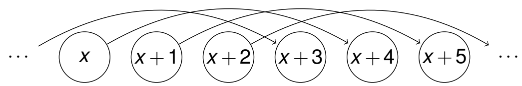
However, if we can make change for x cents, it is not necessarily true that we can make change for $x + 1$ cents.

- We can make change for 10 cents, but not for 11 cents.

If induction is climbing a ladder one step at a time… here we can climb the ladder four steps at a time.

Visualizing Change

Stare at this graph.



We can think of this as four separate ladders:

- $P(0) \implies P(4)$, $P(4) \implies P(8)$, $P(8) \implies P(12)$, …
- $P(1) \implies P(5)$, $P(5) \implies P(9)$, $P(9) \implies P(13)$, …
- $P(2) \implies P(6)$, $P(6) \implies P(10)$, $P(10) \implies P(14)$, …
- $P(3) \implies P(7)$, $P(7) \implies P(11)$, $P(11) \implies P(15)$, …

Idea: If we can make change for four consecutive numbers x, $x+1$, $x+2$, $x+3$, then we can make change for all $n \geq x$.

Making Change

Theorem: Using 4-cent coins and 5-cent coins, we can make change for n cents, where n is any integer which is at least 12.

Proof.

- 12 cents: Use three 4-cent coins.
- 13 cents: Use two 4-cent coins and a 5-cent coin.
- 14 cents: Use a 4-cent coin and two 5-cent coins.
- 15 cents: Use three 5-cent coins.
- Inductively, assume that we can make change for x, $x+1$, $x+2$, and $x+3$, where x is some integer ≥ 12.
- How do we make change for $x+4$? Make change for x, and then add a 4-cent coin. $\square$

Strong Induction

More generally, this introduces the idea that we may need *more* than just $P(n)$ to prove $P(n+1)$.

To prove $\forall n \in \mathbb{N}\ P(n)$, prove:

- $P(0)$;
- $\forall n \in \mathbb{N}\ [(P(0) \wedge P(1) \wedge \cdots \wedge P(n)) \implies P(n+1)]$.

This is called **strong induction**.

Why does this work?

- We proved $P(0)$.
- We proved $P(0)$ and $P(0) \implies P(1)$, so $P(1)$ holds.
- We proved $P(0)$, $P(1)$, and $(P(0) \wedge P(1)) \implies P(2)$, so $P(2)$ holds. (and so on)
- Knock over dominoes, where *all previously knocked down dominoes* help knock over the next domino.

Existence of Prime Factorizations

Theorem: For any natural number $n \geq 2$, we can write n as a product of prime numbers.

Proof.

- Base case: $n = 2$ is itself prime.
- Inductive hypothesis: Let $n \geq 2$ and suppose that n has a prime factorization.
- Inductive step: Either $n+1$ is prime, or $n+1 = ab$ where $a, b \in \mathbb{N}$ with $1 < a, b < n+1$. How do we factor a and b? [1]
- Strong induction: Assume that for all $2 \leq k \leq n$, we know that k has a prime factorization.
- Apply strong inductive hypothesis to a and b to express each as products of primes.
- Thus, $n+1$ is a product of primes. $\square$

[1] Remark: Relating the prime factorization of n with the prime factorization of $n+1$ is an incredibly difficult unsolved problem in number theory.

Strong Induction Is Equivalent to Induction

Strong induction... is a misleading name.

Strong induction implies ordinary induction.

- Ordinary induction is the same as strong induction, except that we *forget* that we proved $P(0), P(1), \ldots, P(n-1)$. We only use $P(n)$ to prove $P(n+1)$.

Ordinary induction implies strong induction.

- Given a sequence of propositions $P(0), P(1), P(2), P(3), \ldots$, define the propositions

$$Q(n) := P(0) \wedge P(1) \wedge \cdots \wedge P(n), \qquad \text{for } n \in \mathbb{N}.$$

- Ordinary induction to prove $\forall n \in \mathbb{N}\ Q(n)$ is *equivalent* to using strong induction to prove $\forall n \in \mathbb{N}\ P(n)$.

Strong Induction

If you do not need strong induction, then just use ordinary (weak) induction.

- Try weak induction first.
- If you need more information, just upgrade to strong induction at no additional cost.

Strong induction is not really a *different* technique from ordinary induction.

Strong induction is a different way to *apply* ordinary induction.

All Horses Are the Same Color

"Theorem": All horses are the same color.

"*Proof*".

- We will use induction on the size of the set of horses.
- Base case: For a set containing one horse, all horses in the set are the same color.
- Inductive hypothesis: Assume that for all sets containing n horses, all horses in the set are the same color.
- Inductive step: Consider a set of $n+1$ horses.
- By the inductive hypothesis, the first n horses are the same color. The last n horses are also the same color.
- Since the first n and last n horses overlap, then all $n+1$ horses are the same color. ♠

Spot the mistake!

Actually, Not All Horses Are the Same Color

The implication $P(1) \implies P(2)$ fails.

- For a set of two horses, the first horse and last horse do **NOT** overlap.

Moral of the story: Be careful!

- Also check the base case!
- The base case is usually easy so it is sometimes ignored.
- This costs you points on the midterm.

Summary

- To prove $\forall n \in \mathbb{N}\ P(n)$, prove:
  1. the base case $P(0)$, and
  2. for all $n \in \mathbb{N}$, assume $P(n)$ and prove $P(n+1)$.
- Domino tilings and moving the hole around:
  - Sometimes *strengthening* the claim makes it easier to prove!
- Strong induction: in the inductive step, assume $P(0), P(1), \ldots, P(n-1)$ in addition to $P(n)$.
- Strong induction is equivalent to ordinary induction.
- All horses are not the same color: you can make mistakes if you are not careful.