Probability Theory Review

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- conditional probability
- random variables and their distributions
- joint distributions, conditional distributions, independence of random variables
- expectation, variance, covariance
- inequalities, confidence intervals, Weak Law of Large Numbers, Central Limit Theorem
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Too much material! Some of these slides will be skipped in lecture, but are included so you can look them over later.

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- Generalized Inclusion-Exclusion: $\mathbb{P}(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n (-1)^{n+1} \sum_{S \subseteq \{1,\dots,n\},\, |S|=i} \mathbb{P}(\bigcap_{j \in S} A_j)$.

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- Note: In probability, $\mathbb{P}(A \cap B)$ is abbreviated as $\mathbb{P}(A, B)$.

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- "Divide and conquer" strategy for calculating $\mathbb{P}(A)$.

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If $B_1, \ldots, B_n$ are possible causes, and A is the effect, then for each $i = 1, \ldots, n$,

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- So if A and B are disjoint, then they are *not* independent (except for edge cases).

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- Abbreviate $\mathbb{P}(\{X = x\})$ as $\mathbb{P}(X = x)$.

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- Then, $\mathbb{P}(X \in [a, b]) = \int_a^b f_X(x)\,\mathrm{d}x$.

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- Give a named distribution with parameters.

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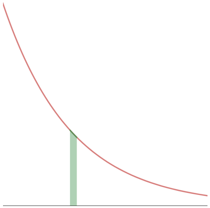

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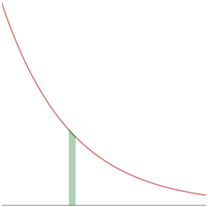

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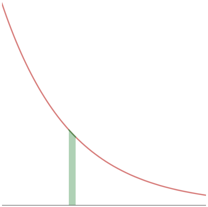

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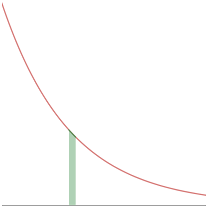

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Multiple Random Variables

When we have two random variables, X and Y, we describe their *joint distribution* via:

- (both discrete) the joint PMF $p_{X,Y}(x,y) := \mathbb{P}(X = x, Y = y)$;
- (both continuous) the joint PDF $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, \mathrm{d}x \, \mathrm{d}y = 1$.

The definitions extend for more than two RVs.

The notation $\mathbb{P}(X = x, Y = y)$ is shorthand for $\mathbb{P}(\{X = x\} \cap \{Y = y\}) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\})$.

How to calculate probabilities: for $A \subseteq \mathbb{R}^2$,

- (discrete) $\mathbb{P}((X,Y) \in A) = \sum_{(x,y) \in A} p_{X,Y}(x,y)$;
- (continuous) $\mathbb{P}((X,Y) \in A) = \iint_A f_{X,Y}(x,y) \, \mathrm{d}x \, \mathrm{d}y$.

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Example where X and Y take on two values each:

	$y = 0$	$y = 1$	p_X
$x = 0$	$p_{X,Y}(0,0) = 0.1$	$p_{X,Y}(0,1) = 0.3$	$p_X(0) = 0.4$
$x = 1$	$p_{X,Y}(1,0) = 0.2$	$p_{X,Y}(1,1) = 0.4$	$p_X(1) = 0.6$
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- From the joint, we can recover the marginals.
- From the marginals of X and Y, we *do not have enough information to recover the joint*.

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Bayes Rule for RVs

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- So if $X_1, \ldots, X_n$ are independent, $\operatorname{var} \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \operatorname{var} X_i$.

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- Example: Pick a permutation of $\{1, \ldots, n\}$ uniformly at random; X is the number of fixed points. For $i = 1, \ldots, n$, X_i indicates if the ith position is fixed, so $X = \sum_{i=1}^n X_i$.

Discrete Distributions Reference

Bernoulli(p) ($p \in [0,1]$):

- Same as Binomial($1, p$).

Binomial(n, p) ($n \in \mathbb{Z}^+$, $p \in [0,1]$):

- PMF: $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x \in \{0, 1, \ldots, n\}$.
- $\mathbb{E}[X] = np$, $\operatorname{var} X = np(1-p)$.

Geometric(p) ($p \in (0,1]$):

- PMF: $p_X(x) = p(1-p)^{x-1}$ for $x \in \mathbb{Z}^+$.
- $\mathbb{E}[X] = 1/p$, $\operatorname{var} X = (1-p)/p^2$.

Poisson(λ) ($\lambda \in (0, \infty)$):

- PMF: $p_X(x) = \exp(-\lambda)\lambda^x / x!$ for $x \in \mathbb{N}$.
- $\mathbb{E}[X] = \lambda$, $\operatorname{var} X = \lambda$.

Continuous Distributions Reference

Uniform($[a,b]$) ($-\infty < a < b < \infty$):

- PDF: $f_X(x) = 1/(b-a)$ for $x \in [a,b]$.
- $\mathbb{E}[X] = (a+b)/2$, $\operatorname{var} X = (b-a)^2/12$.

Exponential(λ) ($\lambda \in (0,\infty)$):

- PDF: $f_X(x) = \lambda \exp(-\lambda x)$ for $x \geq 0$.
- $\mathbb{E}[X] = 1/\lambda$, $\operatorname{var} X = 1/\lambda^2$.

Normal(μ, σ^2) ($\mu \in \mathbb{R}$, $\sigma^2 \geq 0$):

- PDF: $f_X(x) = (2\pi\sigma^2)^{-1/2} \exp[-(x-\mu)^2/(2\sigma^2)]$.
- $\mathbb{E}[X] = \mu$, $\operatorname{var} X = \sigma^2$.

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Coupon collector: Each time you buy a box, you get one of n coupons independently and uniformly at random. Expected number of boxes needed to collect all coupons?

- Let X_i be the time to get the ith new coupon; $X_i \sim \text{Geometric}((n-(i-1))/n)$.
- By linearity of expectation, $\mathbb{E}[X] = n\sum_{i=1}^{n} i^{-1} \approx n \ln n$.

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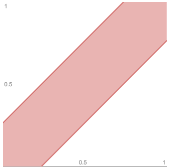

(X, Y) is uniformly distributed on the unit square; the probability is the shaded area $(9/16)$.

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- Calculate: $\mathbb{E}[\bar{X}_n] = \mu$ and $\operatorname{var} \bar{X}_n = \sigma^2/n$.
- Variance shrinks with n. By Chebyshev, $\mathbb{P}(|\bar{X}_n - \mu| \geq \varepsilon) \to 0$ as $n \to \infty$.

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What is the use of the WLLN?

- Suppose we do not know μ.
- If we collect enough samples (n is large), then $\bar{X}_n$ is basically the same thing as μ.

Confidence Intervals

Given n samples and $\delta > 0$, a $1-\delta$ confidence interval for μ is a random interval $(\bar{X}_n - \varepsilon, \bar{X}_n + \varepsilon)$ so that with probability $\geq 1-\delta$, μ lies in the interval.

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Central Limit Theorem

From the WLLN, we know that if we add $X_1, \ldots, X_n$ and divide by n, then we lose almost all information about the distribution: for large n, $(\sum_{i=1}^{n} X_i)/n$ is basically a constant, μ.

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- Scale by $n^{-1/2}$: $Z_n := n^{-1/2}\sum_{i=1}^{n}(X_i - \mu)$.
- Central Limit Theorem (CLT): As $n \to \infty$, Z_n converges in distribution to Normal$(0, \sigma^2)$, in the sense that

$$\mathbb{P}(Z_n \le z) \to \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x.$$

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Markov Property: For all $n \in \mathbb{N}$ and feasible sequences of states $i_0, i_1, \ldots, i_{n-1}, i, j$,

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0)$$

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- "Not depending on n" is *time-homogeneity*.
- We call this quantity $P(i, j)$ and we put it in the (i, j) entry of an $|S| \times |S|$ transition probability matrix P.

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- Again, solve the system of linear equations.

Classification of States

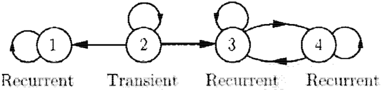

Figure: Figure taken from *Introduction to Probability* by Bertsekas and Tsitsiklis, 2nd edition.

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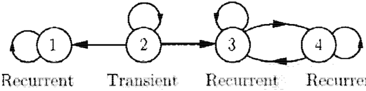

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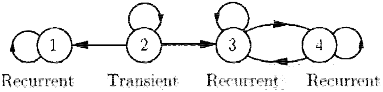

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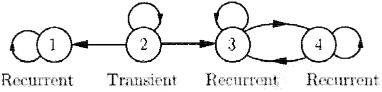

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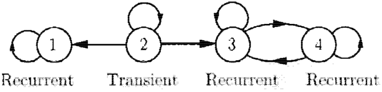

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Do the distributions themselves converge?

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- If the chain is periodic, then convergence can still happen for *some* initial distributions.

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