Discrete Mathematics Review

- logic, proofs
- induction
- graph theory
- modular arithmetic, RSA
- polynomials, error correction
- countability, computability
- counting

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$$\{x \in \mathbb{N} : 2 \leq x \leq 7\} = \{2, 3, 4, 5, 6, 7\}.$$

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- Answer: $(P \wedge \neg Q) \vee (\neg P \wedge Q)$.

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- No; $P(x,y) =$ "x loves y".

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- So, $\mathbb{P}(\bigcup_{i=1}^{n+1} A_i) \leq \sum_{i=1}^{n+1} \mathbb{P}(A_i)$.

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	repeats vertices/edges?	must return to start?
path	no	no
walk	possibly	no
cycle	no	yes
tour	possibly	yes

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- Connected and acyclic; equivalently, connected and has $|V| - 1$ edges.

Graph Theory Results

Handshaking Lemma: $\sum_{v \in V} \deg v = 2|E|$.

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- Hypercubes are bipartite and have Hamiltonian cycles (visit each vertex exactly once).

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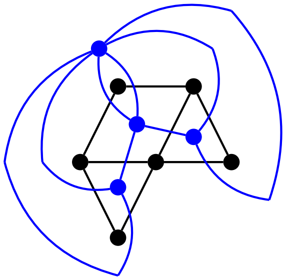

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- Since the vertex has $\leq d_{\max}(G)$ neighbors, color it using color $d_{\max}(G) + 1$.

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So, we let $\mathbb{Z}/m\mathbb{Z} = \{0, 1, \ldots, m-1\}$ be its own number system, with addition and multiplication defined modulo m.

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Chinese Remainder Theorem: For pairwise coprime moduli $m_1, \ldots, m_n$ and fixed $a_1, \ldots, a_n$, the equations $x \equiv a_i \pmod{m_i}$ for $i = 1, \ldots, n$ has a unique solution $x \in \mathbb{Z}/m_1 \cdots m_n\mathbb{Z}$.

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- Security: *Conjectured* to be secure.

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- There is a unique degree $\leq d$ polynomial which passes through any specified $d+1$ distinct points.

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A polynomial is of the form $P(x) = a_d x^d + \cdots + a_1 x + a_0$, where $d \in \mathbb{N}$ is the degree and $a_0, a_1, \ldots, a_d$ are the coefficients.

We look at polynomials over *fields*. Here are fields we care about: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/p\mathbb{Z}$ for p prime.

Facts about polynomials in fields:

- A degree d polynomial has $\leq d$ roots.
- There is a unique degree $\leq d$ polynomial which passes through any specified $d+1$ distinct points.
- Lagrange interpolation: given distinct $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$, then $P(x) := \sum_{i=1}^{d+1} y_i \Delta_i(x)$, where

$$\Delta_i(x) := \frac{\prod_{j \in \{1,\ldots,d+1\} \setminus \{i\}} (x - x_j)}{\prod_{j \in \{1,\ldots,d+1\} \setminus \{i\}} (x_i - x_j)},$$

  is the unique degree $\leq d$ interpolating polynomial.

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- So, $R(x) = r_1 x + r_0$ for some r_0, r_1.
- Since we are in $\mathbb{Z}/p\mathbb{Z}$, there are p choices for r_0 and r_1, so there are p^2 different non-equivalent polynomials.

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- The codeword for the message is $(0, P(0)), (1, P(1)), \ldots, (\ell - 1, P(\ell - 1))$.

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  - This is a system of $n + 2k$ linear equations in the $n + 2k$ unknowns $a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_{n+k-1}$.

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- Alternatively, find an *injection* from an uncountable set (such as $\mathbb{R}$) into the set.

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- Number of subsets of an n-element set? 2^n. Same as the number of length-n bit strings.
- Number of ways to rearrange $\{1, \ldots, n\}$? $n! = \prod_{i=1}^{n} i$.
- Number of k-element subsets of an n-element set? $\binom{n}{k} = \binom{n}{n-k} = n!/[k!(n-k)!]$.
- Number of solutions to $x_1 + \cdots + x_n = k$ in the natural numbers? Throw k unlabeled balls into n labeled bins.
- Stars and bars: the n bins can be represented as $n-1$ "dividers" or bars. Answer: $\binom{n+k-1}{k}$.
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- Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

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- So it equals $\binom{k-1}{n-1}$.

Tomorrow

Review of probability.