Discrete Mathematics Review

- logic, proofs
- induction
- graph theory
- modular arithmetic, RSA
- polynomials, error correction
- countability, computability
- counting

Set Notation

Basic notation: $\in$, $\subseteq$, $\cup$, $\cap$.

- To prove set equality $A = B$, show $A \subseteq B$ and $B \subseteq A$.
- To prove $A \subseteq B$, show that for each $a \in A$, then $a \in B$ also.
- $\{0, 1\}$ is the set containing the two elements 0 and 1.
- $[0, 1]$ is the closed interval containing all x with $0 \leq x \leq 1$.
- $(0, 1)$ is the open interval containing all x with $0 < x < 1$, *or* it is the ordered tuple containing 0 and 1 (context).
- Cartesian product: $A \times B$ is the set of all pairs (a, b) where $a \in A$ and $b \in B$.
  $\{0, 1\} \times \{A, B\} = \{(0, A), (0, B), (1, A), (1, B)\}$.
- We define sets like so: $\{x \in S : \text{conditions on } x\}$. This is the set of all elements in S satisfying the stated conditions.

$$\{x \in \mathbb{N} : 2 \leq x \leq 7\} = \{2, 3, 4, 5, 6, 7\}.$$

Propositional Logic

Language of propositional logic: given propositions P, Q,

- negate a proposition: $\neg P$;
- combine propositions: $P \vee Q$, $P \wedge Q$, $P \implies Q$, $P \iff Q$.

To answer questions in propositional logic, use *truth tables*. Or, use logical equivalences (e.g., De Morgan).

Midterm question: given a truth table

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

can you write an equivalent sentence using P, Q, $\neg$, $\wedge$, $\vee$?

- Answer: $(P \wedge \neg Q) \vee (\neg P \wedge Q)$.

First-Order Logic

First-order logic introduces quantifiers: $\forall$, $\exists$. Now we need more than truth tables; we need semantic proofs.

Recall the intuition:

- $\forall$ is a way to write infinite "AND"s;
- $\exists$ is a way to write infinite "OR"s.

Recall De Morgan: $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$ and $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$.

Question for review: is $\forall x\, \exists y\, P(x,y) \equiv \exists y\, \forall x\, P(x,y)$?

- No; $P(x,y) =$ "x loves y".

Induction

Principle of induction: To prove a statement $\forall n \in \mathbb{N},\ P(n)$,

- (base case) prove $P(0)$;
- (inductive step) prove $\forall n \in \mathbb{N},\ P(n) \implies P(n+1)$.

Union bound: for events A, B, $\mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$.

For positive integers n and events $A_1, \dots, A_n$, prove $\mathbb{P}(\bigcup_{i=1}^{n} A_i) \le \sum_{i=1}^{n} \mathbb{P}(A_i)$?

- Base cases: $n = 1$ obvious; $n = 2$ is given above.
- Inductive step: Assume $P(n)$. Prove $P(n+1)$.
- Let $A_1, \dots, A_{n+1}$ be events. Then, $\mathbb{P}(\bigcup_{i=1}^{n+1} A_i) = \mathbb{P}((\bigcup_{i=1}^{n} A_i) \cup A_{n+1}) \le \mathbb{P}(\bigcup_{i=1}^{n} A_i) + \mathbb{P}(A_{n+1})$.
- Apply inductive hypothesis. $\mathbb{P}(\bigcup_{i=1}^{n} A_i) \le \sum_{i=1}^{n} \mathbb{P}(A_i)$.
- So, $\mathbb{P}(\bigcup_{i=1}^{n+1} A_i) \le \sum_{i=1}^{n+1} \mathbb{P}(A_i)$.

Other Forms of Induction

Strengthening the inductive hypothesis: Instead of proving $\forall n \in \mathbb{N},\ P(n)$, prove $\forall n \in \mathbb{N},\ Q(n)$, where $Q(n)$ implies $P(n)$.

- Try tiling a $2^n \times 2^n$ grid with the upper right corner missing using L-shaped tiles. Get stuck at the inductive step!
- Instead, tile a $2^n \times 2^n$ grid with *any* square missing.
- Use this when your inductive hypothesis does not give you enough information.

Strong induction: During inductive step, you can use $P(0), P(1), \ldots, P(n)$ to help you prove $P(n+1)$.

- This is needed when you reduce, not just to the *previous* case $P(n)$, but to an even smaller case.

Well ordering principle: Every non-empty subset of $\mathbb{N}$ has a least element.

- Consider the *least counterexample*; prove there is an even smaller counterexample!

Graph Theory

A graph is a set of vertices V and a set of edges E.

Recall definitions: degree, connectedness. Types of graphs: trees, forests, planar, bipartite, complete, hypercubes.

Confusing terminology: paths, walks, cycles, tours?

	repeats vertices/edges?	must return to start?
path	no	no
walk	possibly	no
cycle	no	yes
tour	possibly	yes

Graph Theory Results

Handshaking Lemma: $\sum_{v \in V} \deg v = 2|E|$.

- Example: For K_n, $n(n-1) = 2|E|$, so $|E| = n(n-1)/2 = \binom{n}{2}$.

Eulerian tours: Use every edge exactly once.

- An Eulerian tour exists if and only if the graph is connected and every vertex has even degree.

Trees:

- Connected and acyclic; equivalently, connected and has $|V| - 1$ edges. Smallest connected graphs!
- Trees are planar.

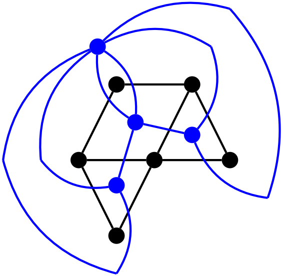
Hypercubes:

- Vertices consist of length-d bit strings; two vertices are adjacent iff they differ in one bit.
- Hypercubes are bipartite and have Hamiltonian cycles (visit each vertex exactly once).

Planarity

Planarity: can be drawn on a plane without edge crossings.

- We only discussed connected planar graphs.
- Euler's formula: $v + f = e + 2$.
- For $|V| \geq 3$, this gives $e \leq 3v - 6$.
- Important non-planar graphs: $K_{3,3}$, K_5.
- Every planar graph has a dual planar graph.



Graph Induction

A graph can be colored with $d_{\max} + 1$ colors, where $d_{\max}$ is the maximum degree of the graph.

- Use induction on the number of vertices.
- Base case: A graph with one vertex only needs one color.
- Inductive hypothesis: Any graph H with n vertices can be colored with $d_{\max}(H) + 1$ colors.
- Consider a graph G with $n + 1$ vertices. Remove a vertex and its associated edges from G to form a graph G'.
- G' has n vertices, and $d_{\max}(G') \leq d_{\max}(G)$. Apply inductive hypothesis to color G' with $\leq d_{\max}(G) + 1$ colors.
- Add the vertex and edges back to G' to form G.
- Since the vertex has $\leq d_{\max}(G)$ neighbors, color it using color $d_{\max}(G) + 1$.

Modular Arithmetic

For a positive integer $m \geq 2$, say two numbers $x, y \in \mathbb{Z}$ are equivalent modulo m, $x \equiv y \pmod{m}$, if $m \mid x - y$.

If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then we can add and multiply these equations as normal:

$$a + c \equiv b + d \pmod{m}, \qquad ac \equiv bd \pmod{m}.$$

Every $x \in \mathbb{Z}$ is equivalent to exactly one of $\{0, 1, \ldots, m-1\}$. So, we let $\mathbb{Z}/m\mathbb{Z} = \{0, 1, \ldots, m-1\}$ be its own number system, with addition and multiplication defined modulo m.

Multiplicative Inverses

For $a \in \mathbb{Z}/m\mathbb{Z}$, the following are equivalent:

- a has a multiplicative inverse in $\mathbb{Z}/m\mathbb{Z}$, i.e., there exists $x \in \mathbb{Z}/m\mathbb{Z}$ so that $ax = 1$.
- $f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ defined by $f(x) := ax$ is a bijection.
- $\gcd(a, m) = 1$.

If a satisfies the three statements above, then we say $a \in (\mathbb{Z}/m\mathbb{Z})^{\times}$.

When p is prime, then $(\mathbb{Z}/p\mathbb{Z})^{\times} = \{1, \ldots, p-1\}$. Every non-zero element has a multiplicative inverse.

Extended Euclid's algorithm: given $a, m \in \mathbb{Z}$, $m \neq 0$, output $x, y \in \mathbb{Z}$ such that $ax + my = \gcd(a, m)$.

- For $a \in (\mathbb{Z}/m\mathbb{Z})^{\times}$, this gives $ax + my = 1$. So, x is the multiplicative inverse of a in $\mathbb{Z}/m\mathbb{Z}$.

Modular Arithmetic Results

Repeated squaring (or fast modular exponentiation): Calculate $a^b \bmod m$ fast!

- Try $3^{60} \bmod 13$.
- Square the base, halve the exponent. $3^{60} = 9^{30} = 81^{15}$.
- Reduce the base: $81^{15} = 3^{15}$.
- For an odd exponent, pull out one power.
  $3^{15} = 3 \cdot 3^{14} = 3 \cdot 9^7 = \cdots$

Fermat's Little Theorem: For p prime and $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, one has $a^{p-1} \equiv 1 \pmod p$.

- Or, for all $a \in \mathbb{Z}/p\mathbb{Z}$, $a^p \equiv a \pmod p$.

Chinese Remainder Theorem: For pairwise coprime moduli $m_1, \ldots, m_n$ and fixed $a_1, \ldots, a_n$, the equations $x \equiv a_i \pmod{m_i}$ for $i = 1, \ldots, n$ has a unique solution $x \in \mathbb{Z}/m_1 \cdots m_n\mathbb{Z}$.

RSA

RSA public-key cryptosystem:

- Generate two distinct large primes, p and q. Let $N := pq$.
- Pick a public key $e \in (\mathbb{Z}/(p-1)(q-1)\mathbb{Z})^\times$. The private key d is the inverse of e in $\mathbb{Z}/(p-1)(q-1)\mathbb{Z}$.
- Public information: (N, e). Only the receiver knows d.
- For a message m, encrypt using $E(m) = m^e \pmod N$ and then send. Receiver decrypts using $D(c) = c^d \pmod N$.

RSA details:

- Correctness: Proof uses Fermat's Little Theorem.
- Efficiency: Repeated squaring, extended Euclid, Prime Number Theorem, primality tests.
- Security: *Conjectured* to be secure.

Polynomials

A polynomial is of the form $P(x) = a_d x^d + \cdots + a_1 x + a_0$, where $d \in \mathbb{N}$ is the degree and $a_0, a_1, \ldots, a_d$ are the coefficients.

We look at polynomials over *fields*. Here are fields we care about: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/p\mathbb{Z}$ for p prime.

Facts about polynomials in fields:

- A degree d polynomial has $\leq d$ roots.
- There is a unique degree $\leq d$ polynomial which passes through any specified $d+1$ distinct points.
- Lagrange interpolation: given distinct $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$, then $P(x) := \sum_{i=1}^{d+1} y_i \Delta_i(x)$, where

$$\Delta_i(x) := \frac{\prod_{j \in \{1,\ldots,d+1\} \setminus \{i\}} (x - x_j)}{\prod_{j \in \{1,\ldots,d+1\} \setminus \{i\}} (x_i - x_j)},$$

  is the unique degree $\leq d$ interpolating polynomial.

Midterm Question

Polynomials P and Q (over $\mathbb{Z}/p\mathbb{Z}$) are equivalent modulo x^2+1 if $P(x)-Q(x) = K(x)(x^2+1)$ for some polynomial K.

- Similar to the definition of modular equivalence!

How many polynomials can you put into a set so that no two of them are equivalent modulo x^2+1?

- First step: How many numbers are in $\mathbb{Z}/m\mathbb{Z}$?
- For $x \in \mathbb{Z}$, Division Algorithm gives $x = qm + r$ where $q \in \mathbb{Z}$ and $r \in \{0, 1, \ldots, m-1\}$. So, $x \equiv r \pmod{m}$.
- Similarly, $P(x) = Q(x)(x^2+1) + R(x)$ for polynomials Q and R, where $\deg R < 2$.
- So, $R(x) = r_1 x + r_0$ for some r_0, r_1.
- Since we are in $\mathbb{Z}/p\mathbb{Z}$, there are p choices for r_0 and r_1, so there are p^2 different non-equivalent polynomials.

Applications of Polynomials

Shamir's secret sharing:

- If k officers get together, they know the secret $s \in \mathbb{Z}/p\mathbb{Z}$. If $\leq k-1$ officers get together, they learn nothing.
- Define $P(x) := s_{k-1}x^{k-1} + \cdots + s_1 x + s$, where $s_1, \ldots, s_{k-1}$ are chosen randomly.
- Give each officer an evaluation of the polynomial.

Reed-Solomon codes:

- Given a message $(m_0, m_1, \ldots, m_{n-1})$, encode it as a polynomial $P(x) = m_{n-1}x^{n-1} + \cdots + m_1 x + m_0$.
- Encode the message as a codeword of length ℓ.
- The codeword for the message is $(0, P(0)), (1, P(1)), \ldots, (\ell - 1, P(\ell - 1))$.

Reed-Solomon Error Correction

- A Reed-Solomon code with codeword length $\ell = n + k$ can recover the message if $\leq k$ packets are *erased*.
- A Reed-Solomon code with codeword length $\ell = n + 2k$ can recover the message if $\leq k$ packets are *corrupted*.
  - This code has minimum pairwise Hamming distance $2k + 1$ $\implies$ correct k general errors.
- Berlekamp-Welch: An efficient decoding scheme for Reed-Solomon codes under corruption errors.
  - If errors are at $e_1, \dots, e_k$, define $E(x) = \prod_{i=1}^{k}(x - e_i) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$.
  - Define $Q(x) = P(x)E(x) = b_{n+k-1}x^{n+k-1} + \cdots + b_1 x + b_0$.
  - Key Lemma: If $R_0, R_1, \dots, R_{n+2k-1}$ are the received packets, then $R_i E(i) = P(i)E(i)$ for $i = 0, 1, \dots, n + 2k - 1$.
  - This is a system of $n + 2k$ linear equations in the $n + 2k$ unknowns $a_0, a_1, \dots, a_{k-1}, b_0, b_1, \dots, b_{n+k-1}$.

Countability

A set S is countable if there is an injection $S \to \mathbb{N}$.

- Countable sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, $\mathbb{Q}$. All finite-length strings from a countably infinite alphabet.
- How to show a set is countable: put its elements into a list!

A set S is uncountable if it is not countable.

- Examples: $\mathbb{R}$, infinite-length bit strings.
- How to show a set is uncountable: Cantor diagonalization.

$$\begin{matrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

What infinite-length bit string is not in the list? $101\ldots$

- Alternatively, find an *injection* from an uncountable set (such as $\mathbb{R}$) into the set.

Computability

Not all functions can be computed.

- Link to countability: computer programs are countably infinite, but functions $\mathbb{N} \to \{0,1\}$ are uncountable.
- `TestHalt` takes two arguments, a program *P* and an input *x*, and returns 1 iff $P(x)$ halts; 0 otherwise.
- Then, $\texttt{TestHalt} : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ is an *explicit* function which cannot be computed.

Reductions:

- To show that *P* is uncomputable, *assume P exists*. But, *do not assume how P is implemented*.
- Example: To show that `TestHalt` is uncomputable, *do not assume that `TestHalt` must actually run $P(x)$*.
- Then, use the power of *P* to define `TestHalt`, which you know is impossible.
- Therefore, *P* cannot exist.

Counting

- Number of subsets of an n-element set? 2^n. Same as the number of length-n bit strings.
- Number of ways to rearrange $\{1, \ldots, n\}$? $n! = \prod_{i=1}^n i$.
- Number of k-element subsets of an n-element set? $\binom{n}{k} = \binom{n}{n-k} = n!/[k!(n-k)!]$.
- Number of solutions to $x_1 + \cdots + x_n = k$ in the natural numbers? Throw k unlabeled balls into n labeled bins.
- Stars and bars: the n bins can be represented as $n-1$ "dividers" or bars. Answer: $\binom{n+k-1}{k}$.
- If A and B are disjoint, what is $|A \cup B|$? Answer: $|A| + |B|$.
- What if A and B are not disjoint? Inclusion-Exclusion: $|A| + |B| - |A \cap B|$.
- Binomial Theorem: $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$.

Combinatorial Proofs

Prove an equation involving combinatorial terms by showing that both sides count the same objects.

Midterm: For $k \geq n$,
$\binom{k-1}{n-1} = |\{(x_1, \ldots, x_n) \in \mathbb{N}^+ : x_1 + \cdots + x_n = k\}|$.

- On the RHS, since $x_1 + \cdots + x_n = k$, think of splitting up k things into n chunks of size ≥ 1 each.
- How many ways are there to create these partitions?
- This is like placing $n-1$ dividers among the k objects...
- So it equals $\binom{k-1}{n-1}$.

Tomorrow

Review of probability.