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Today: We will learn strategies for writing valid proofs.

Counterintuitive Mathematics

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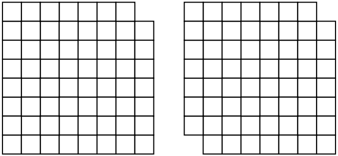
$$\int_0^\infty \frac{\sin x}{x}\,\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/15)}{x/15}\,dx$$

$$= \frac{\pi}{2} - \frac{6879714958723010531\pi}{935615849440640907310521750000}.$$

Chessboard Tilings



Can you tile the first grid using 1×2 tiles? 

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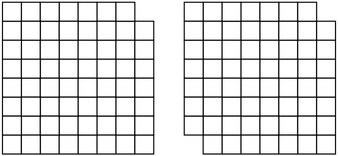

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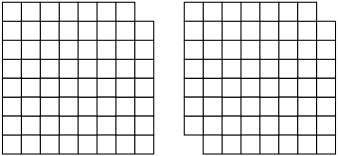

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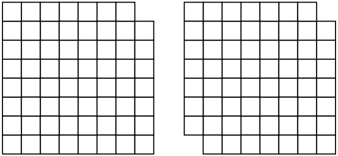

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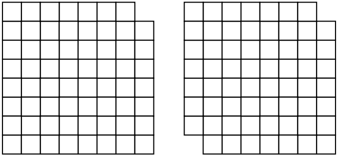

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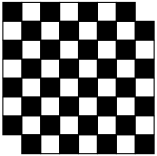

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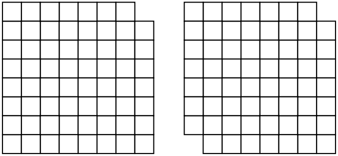

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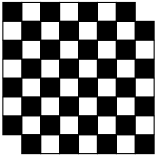

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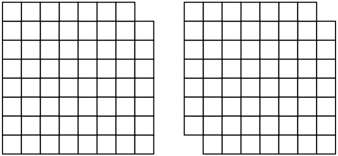

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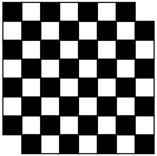
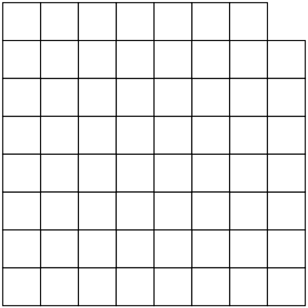


- No. The board has more black squares than white squares.

Preview



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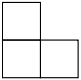

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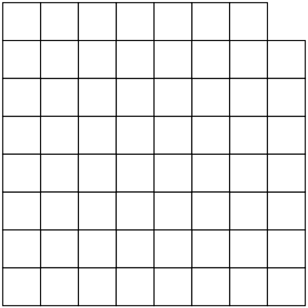

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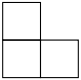

Next lecture!

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Direct Proof: Example

Background: Given $a, b \in \mathbb{Z}$, we say that *a* **divides** *b*, written $a \mid b$, if there exists an integer $d \in \mathbb{Z}$ such that $ad = b$.[1]

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- By definition, $ac \mid bc$, which is Q. □

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Similarly, $c \mid a - b$. In fact, for any $x, y \in \mathbb{Z}$, we have $c \mid xa + yb$.

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- Or... when $\neg P$ is easier to prove than Q.

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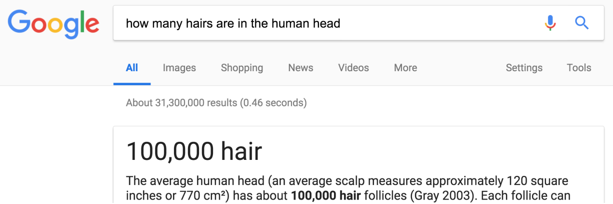
- Contrapositive: If no hole has more than one pigeon, the number of pigeons is at most the number of holes.
- *Proof.* Every hole has zero or one pigeons, so the number of holes is at least as big as the number of pigeons. □

Application of Pigeonhole Principle



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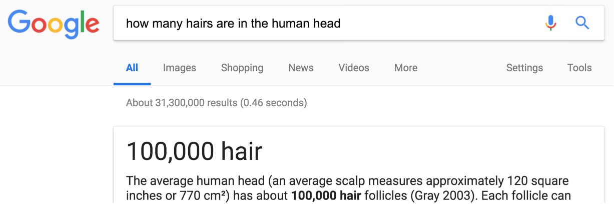

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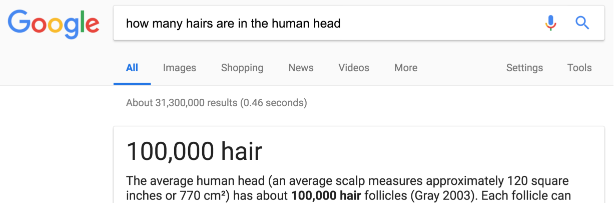

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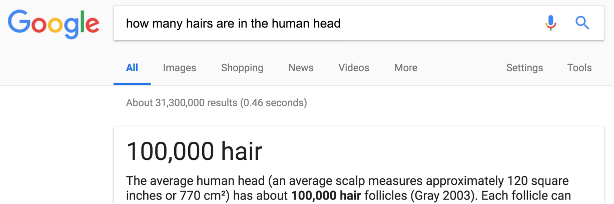

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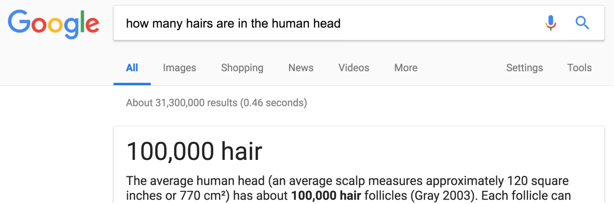

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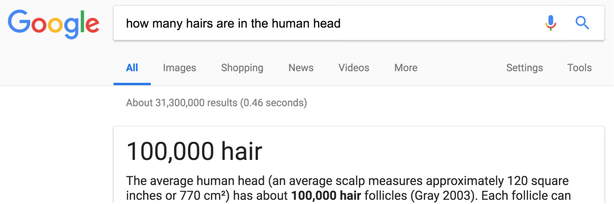

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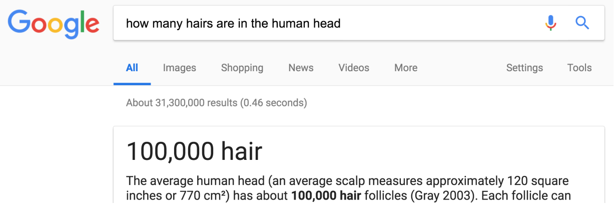

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- The people of SF are pigeons.
- The number of head hairs that a person has is a “box”.
- Conclusion: There are two people in SF who have the same number of hairs on their heads.

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Notice the use of the contrapositive. In fact, proof by contraposition and proof by contradiction are not very different.

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Remark: In higher mathematics, proofs are usually phrased via contradiction rather than contraposition.

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- Assume, for the sake of contradiction, that $\sqrt{2}$ is rational.
- Then, let $p, q \in \mathbb{Z}$ be such that $\sqrt{2} = p/q$. Let p and q be in lowest terms.
- Square it! $2 = p^2/q^2$.
- If p and q share no common factors, then neither do p^2 and q^2... but $p^2 = 2q^2$, so q^2 divides p^2. Contradiction. □

Proof by Contradiction: Infinitude of Primes

Background: A **prime number** is a natural number, larger than 1, whose only positive divisors are 1 and itself.

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Is $p_1 \cdots p_n + 1$ prime? *Not necessarily.*
$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 1 = 19 \cdot 97 \cdot 277$.

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- Conclude P.

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When using proof by cases, save work by eliminating unnecessary cases.

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WLOG means we are considering a special case, but *from this special case we can recover the general case easily*.

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- The cases are exhaustive. $\square$

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(But how do we know e and $\ln 2$ are irrational?)

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Moral: Do not assume what you are trying to prove!

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4. How rigorous should my proof be? Rule of thumb: good enough to convince your skeptical classmate.

Summary

- Direct proof
- Proof by contraposition, proof by contradiction
- Proof by cases
- Pigeonhole principle: More pigeons than holes implies that at least one hole has multiple pigeons.
- Proofs can be non-constructive.
- You learned classical proofs: irrationality of $\sqrt{2}$, infinitude of primes...