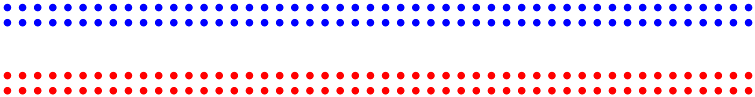
Counting ∞



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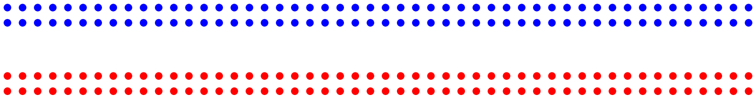

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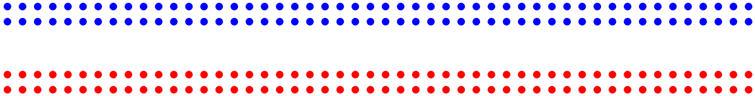

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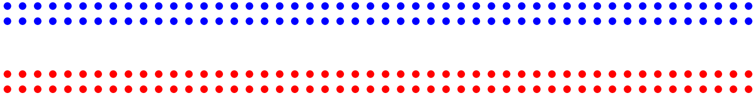

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Today: We count to ∞ and beyond.

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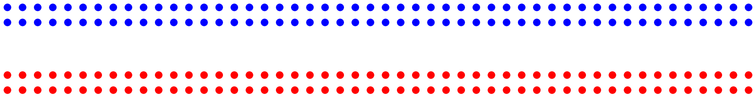
A bijection "rearranges" the elements of A to form B.

Counting Infinite Sets



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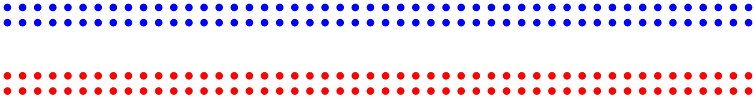

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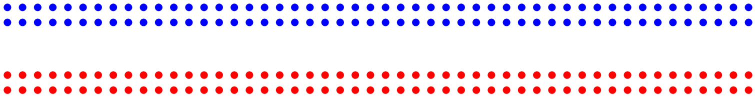

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To count infinities, we will take the definition of “same size” to be “there exists a bijection between the sets”.

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- What else is countable?

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Adding a countably infinite number of elements to $\mathbb{N}$ does not change its size.

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$$2, 3, 5, 7, 11, \ldots$$

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Here are some countably infinite sets.

- $\mathbb{N}$. $\mathbb{N} \cup \{-1\}$. $\mathbb{Z}$.
- The set of even numbers. The set of odd numbers.
- The set of prime numbers.

Why is the set of prime numbers countably infinite? It is infinite (we proved this). But we can *list* them.

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A set whose elements can be listed is countable.

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$$0, 1, 2, 3, \ldots, -1, -2, -3, \ldots$$

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Be careful with "$\ldots$" in the middle of your listing.

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Adding a countably infinite number of countable infinities to $\mathbb{N}$ does not change its size.

The Formal Injection

We found an injection

$$f : \{\text{prime numbers}\} \times \{1, 2, 3, \ldots\} \to \{\text{odd numbers}\}$$

given by $f(p, i) = p^i$-th odd number.

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Why? There are bijections

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so we get an injection $g(m,n) = f_3^{-1}(f(f_1(m), f_2(n)))$.

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Fact: If $f : A \to B$ and $g : B \to C$ are bijections, then so is $g \circ f$.

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Bijections compose.

Exercise: If there are bijections $f : A \to A'$ and $g : B \to B'$, then $h(a,b) = (f(a), g(b))$ is a bijection $A \times B \to A' \times B'$.

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Fact: If $f : A \to B$ is a bijection, and there are bijections $f_1 : A \to A'$ and $f_2 : B \to B'$, then there is a bijection $g : A' \to B'$.

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To show that A has the same size as $\mathbb{N}$, we can show that A has the same size as A', where A' has the same size as $\mathbb{N}$.

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- We found an injection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. So, $\mathbb{N} \times \mathbb{N}$ is countable.
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- On the other hand, $\mathbb{Q}$ is infinite, so $\mathbb{Q}$ is countably infinite.

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Principle: To show that a set A is countable, we only need to find an injection from A into a countable set.

Interleaving Argument

Suppose that A is a countable alphabet. Consider the set of all *finite* strings whose symbols come from A.

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- Continue forever.

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- Step 3: List all strings of length ≤ 3 using symbols from $\{a_1, a_2, a_3\}$.
- Continue forever. This exhaustively lists the members of the set. □

Polynomials with Rational Coefficients

Consider the set of polynomials with rational coefficients.

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For a polynomial, e.g., $P(x) = (2/3)x^4 - 2x^2 + (1/10)x + 9$, think of it as a string: $(2/3, 0, -2, 1/10, 9)$.

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- Change all 8s to 1s and change all other numbers to 8s.

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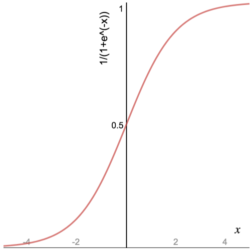

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Example of a function $f : \{0,1,2\} \to \mathscr{P}(\{0,1,2\})$:

$$f(0) = \{1,2\}, \qquad f(1) = \{1\}, \qquad f(2) = \varnothing.$$

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- Conclusion: No x gets mapped to A. So f cannot be surjective. □

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- Example: $\{2, 3, 4\} \equiv (0, 0, 1, 1, 1, 0, 0, 0, \dots)$

In fact, since the numbers in $[0, 1]$ can be written as infinite-length bit strings (binary expansion), there is a bijection

$$f : [0, 1] \to \mathscr{P}(\mathbb{N}).$$

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s_0	N	N	N	$\cdots$
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The previous proof is also a proof by diagonalization!

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Are there cardinalities between $\mathbb{N}$ and $\mathbb{R}$? (**Continuum Hypothesis**) Not provable/disprovable from our axioms!

Summary

- A set is countable if there is an injection into $\mathbb{N}$.
- Countable sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, prime numbers, finite-length strings from a countable alphabet.
- Cantor introduced a diagonalization argument. We proved that $[0, 1]$ is uncountable.
- Cantor-Schröder-Bernstein Theorem: If there is an injection both ways, there is a bijection.
- The power set is strictly larger than the original set!