Note 5 Supplement: Planar Duality

Computer Science 70
University of California, Berkeley

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1 Planar Graphs

A graph $G = (V, E)$ is a **planar graph** if it can be drawn in the plane with no edge crossings. We assume that G is finite (it has finitely many vertices and edges) and undirected. We will *not* assume that G is a simple graph, however, because we will need to discuss planar graphs which possibly contain *multiple* edges between pairs of vertices, or even an edge from a vertex to itself (a self-loop). Finally, we will only be discussing *connected* graphs.

There are many ways to embed the graph G into a plane with no edge crossings (we call such an embedding a *planar drawing*). For the rest of this note, we will fix some planar drawing of G and refer to it.

Perhaps the simplest examples of planar graphs are trees, i.e., connected acyclic graphs.

Theorem 1. *Trees are planar graphs.*

Proof. Proceed by induction on the number of vertices. If the tree has one vertex, then it is certainly planar. Otherwise, consider a tree with at least two vertices and assume that all trees with fewer vertices are planar.

Remove a leaf v and its associated edge $\{u, v\}$ from the tree. The resulting graph is a tree, so by the inductive hypothesis, it has a planar drawing. Now "zoom in" on the vertex u in this planar drawing (this requires you to visualize the tree in your head). If we "zoom in" close enough, then we will only see the vertex u and the edges leaving the vertex u. Since the vertex u is only connected to finitely many other vertices, we can place the vertex v in an empty spot in between some of the edges leaving u and add back the edge $\{u, v\}$, yielding a planar drawing of our original tree. □

A **face** is a connected region of the plane (in the planar drawing). Since the graph is finite, one of the faces (the face which lies "outside" of the edges in the planar drawing) is infinite.

We will use the following notation:

- $v := |V|$ is the number of vertices in G;
- $e := |E|$ is the number of edges in G;
- f is the number of faces in G.

There is a relationship between these quantities for planar graphs:

Theorem 2 (Euler's Formula)**.** *For any connected planar graph, $v+f = e+2$.*

One can prove Euler's Formula by induction. Instead, we will present a different proof which uses a deeply fascinating concept called *planar duality*.

2 Planar Duality

A **planar dual** of G, called G^*, is a planar graph whose vertices are the faces of G and whose edges correspond to the edges of G. For this, a picture is worth much more than the formal definition (see Figure 1).

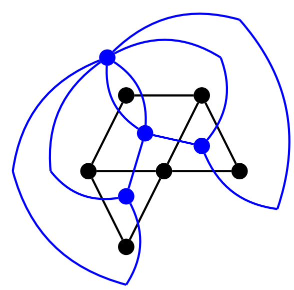

Figure 1: A planar graph (in black) is shown, along with a planar dual (in blue) vertices of he dual planar graph correspond to the faces of the original planar graph. Each edge of the original planar graph has a face on each of its two sides (the two faces are not necessarily distinct); the corresponding edge in the dual planar graph connects these two faces.

For a planar graph G, there may be multiple ways to draw a dual planar graph, so technically we should speak about "*a* planar dual" rather than "*the* planar dual". For our purposes, it does not really matter what planar dual we choose, so we will not worry about this any further.

There is an intimate relationship between a planar graph and its dual. First of all, when we speak of *cycles*, we mean simple cycles, i.e., cycles that do not repeat edges and do not repeat vertices (except for the fact that a cycle starts and ends at the same vertex). A cycle encloses one or more faces of the graph. A **cut** in the graph is a set of edges in the graph which, if removed, disconnects a set of vertices from the rest of the graph (in this definition, the set of vertices S is assumed to be non-empty, and not equal to V itself). When we speak of cuts, however, we mainly refer to *minimal cuts*, which are cuts such that no proper subset of edges is also a cut.

Theorem 3 (Cycle-Cut Duality)**.** *Every cycle in G corresponds to a unique cut in G^*.*

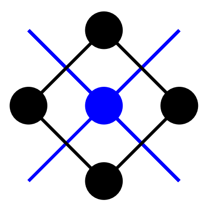

Figure 2: A cycle in G corresponds to a cut in G^*.

Proof. See Figure 2 for a visualization.

A cycle in G encloses one or more faces of G, i.e., a cycle encloses one or more vertices of G^*. Thus, the dual edges to the cycle form a cut in G^* which separate these dual vertices from the rest of the dual graph.

Conversely, a minimal cut in G^* consists of the set of edges with one endpoint in S and one endpoint in $V \setminus S$, where S is some proper non-empty subset of vertices. The corresponding edges in G must enclose either the faces corresponding to S or the faces corresponding to $V \setminus S$, so these edges must be a simple cycle in G. □

As a special case of Cycle-Cut Duality, we have the following:

Theorem 4. *G is a planar dual of G^*.*

In other words, the graph G is a dual of its dual.

Proof of Theorem 4. First, for any vertex in G, the edges leaving the vertex correspond to dual edges which define a face in G^*; thus, vertices in G correspond to faces in G^*.

Since we already know that the edges of G and G^* correspond to each other, then G is a planar dual of G^*. □

Since G is a dual of its dual, then Cycle-Cut Duality also implies that each minimal cut in G corresponds to a unique simple cycle in G^*.

The last piece of information that we will need is to relate cuts and the connectedness of a graph.

Theorem 5. *Let E' be a subset of edges in a graph $G = (V, E)$. Then, the graph (V, E') is connected if and only if the remaining edges $E \setminus E'$ do not contain any cuts.*

Proof. If $E \setminus E'$ does not contain any cuts, then removing the edges in $E \setminus E'$ does not disconnect any subset of vertices, i.e., the graph (V, E') is connected.

Conversely, if $E \setminus E'$ contains a cut, then there is a subset S of vertices for which the removal of the edges in the cut disconnects S from the rest of the vertices. Then, the graph (V, E') is not connected because there is no path from a vertex in S to a vertex in $V \setminus S$. □

Now, in our planar graph G, we find a **spanning tree**, that is, a subset E' of edges in G such that (V, E') is a tree. We can find this spanning tree because G is connected. See Figure 3 for a picture of a spanning tree of the planar graph in Figure 1.

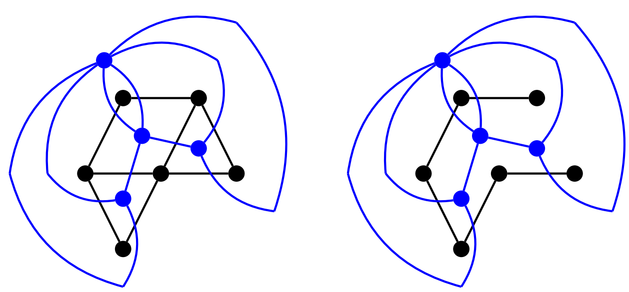

Figure 3: In the figure on the right, the remaining black edges form a spanning tree of the planar graph in Figure 1.

Next, we look at the edges in G^* which are *not* dual edges to the spanning tree (V, E'). Thus, these edges are dual to $E \setminus E'$. Call these edges $\overline{E'}$. See Figure 4 for a picture of these edges.

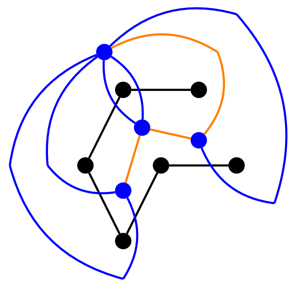

Figure 4: The orange edges are the edges in G^* which are *not* the dual edges of the spanning tree in G.

Since (V, E') is connected, then by Theorem 5, the remaining set of edges $E \setminus E'$ does not contain any cuts. By Cycle-Cut Duality (Theorem 3), the corresponding dual edges $\overline{E'}$ are acyclic.

On the other hand, since (V, E') is acyclic, then again by Cycle-Cut Duality (Theorem 3), the dual edges to E' do not have any cuts; thus, the remaining set of edges $\overline{E'}$ form a connected subgraph in G^* by Theorem 5.

Hence, we have found that the set of edges $\overline{E'}$ is connected and acyclic, so they form a spanning tree of G^*. We call this the **dual spanning tree** to the spanning tree (V, E').

Now we are ready to prove Euler's Formula:

Proof of Theorem 2. Consider a spanning tree (V, E') of G and the edges $\overline{E'}$ which form the dual spanning tree in G^*. Since the number of edges in a tree is one less than the number of vertices, then $|E'| = v - 1$ and $|\overline{E'}| = f - 1$. However, since the edges in $\overline{E'}$ are dual to $E \setminus E'$, then the total number of edges in E' and $\overline{E'}$ is equal to E, so we get $e = (v - 1) + (f - 1)$. Rearranging this equation yields Euler's Formula. □

As a closing note, if you prove a theorem about planar graphs, then you can apply the theorem to a planar dual to see if you can get another "dual" theorem out. For example, we have proved the following:

Lemma 1 (Handshaking Lemma). $\sum_{v \in V} \deg v = 2|E|$.

Now, applying the Handshaking Lemma to a planar dual of $G = (V, E)$, we get $\sum_{f \in F} \deg f = 2|E|$, where F is the set of faces in G. Here, $\deg f$ is the degree of the vertex v in the planar dual, which is equal to the number of *sides* of the face in G. We have the statement that $2|E|$ is the total number of sides in G.

On the other hand, if the planar graph has at least three vertices, then each face has at least three sides. So, $2e \geq 3f$, and by plugging this into Euler's Formula (Theorem 2) and rearranging, we get the inequality:

Corollary 1. *If G has at least three vertices, then $e \leq 3v - 6$.*

This last result can be used to prove the non-planarity of K_5, the complete graph on five vertices.