Note 10 Supplement: Cantor-Schröder-Bernstein Theorem

Computer Science 70
University of California, Berkeley

Summer 2018

This note is optional. Please read on only if you are interested.

Can you explicitly construct a bijection between $(0,1]$ and $[0,1]$? It is not so easy to find a bijection directly, but with the help of the following theorem, it suffices to find an injection in each direction.

Theorem 1 (Cantor-Schröder-Bernstein)**.** *If $f : X \to Y$ and $g : Y \to X$ are injections (one-to-one), then there exists a bijection $\varphi : X \to Y$.*

Proof. The proof is constructive. We start with a point $x \in X$, and we attempt to apply the function g^{-1}, i.e. we try to compute $g^{-1}(x)$. This is only possible if x lies in the range of g, but let us accept this for now. The point $g^{-1}(x)$ lies in the set Y, and now we can try to compute $f^{-1}(g^{-1}(x))$. Again, this may or may not be possible, depending on whether $g^{-1}(x)$ lies in the range of f, but we try anyway. If we succeed, then we can try to compute $g^{-1}(f^{-1}(g^{-1}(x)))$ and so forth. Now, for every $x \in X$, we apply this procedure, and there are three possible outcomes:

1. This procedure terminates at some point, because we are unable to apply g^{-1}. If so, we put x in the set X_X.

2. This procedure terminates at some point, because we are unable to apply f^{-1}. If so, we put x in the set X_Y.

3. This procedure never terminates. If so, we put x in the set X_∞.

Thus, X is the disjoint union of the three sets above, $X = X_X \cup X_Y \cup X_\infty$. Now, we apply the same trick to partition Y as well. Explicitly, fix some $y \in Y$ and attempt to apply f^{-1} to x; if we succeed then we try to apply g^{-1} to $f^{-1}(y)$, and if we succeed we attempt to apply f^{-1} to $g^{-1}(f^{-1}(y))$, etc. Again, we split Y into three cases:

1. Define Y_X to be the set of $y \in Y$ for which the procedure terminates, because we cannot apply g^{-1} anymore.
2. Define Y_Y to be the set of $y \in Y$ for which the procedure terminates, because we cannot apply f^{-1} anymore.
3. Define Y_∞ to be the set of $y \in Y$ for which the procedure never terminates.

We can write Y as a disjoint union as well: $Y = Y_X \cup Y_Y \cup Y_\infty$. Now, our plan is to construct the bijection φ in the following way:

$$\varphi(x) = \begin{cases} f(x), & x \in X_X \\ g^{-1}(x), & x \in X_Y \\ f(x), & x \in X_\infty \end{cases}$$

The bijection above is actually composed of three different bijections:

$$X_X \xrightarrow{f} Y_X$$
$$X_Y \xrightarrow{g^{-1}} Y_Y$$
$$X_\infty \xrightarrow{f} Y_\infty$$

Our task now is to verify the above bijections. First, a bit of terminology to simplify the explanation: if we apply the procedure for partitioning X to a point $x \in X$, we say that we are applying the X-procedure to x. (Here, "applying the procedure" simply means that we apply g^{-1} and f^{-1} alternately until the procedure terminates, or does not terminate at all.) Similarly, if we apply the procedure for partitioning Y to a point $y \in Y$, we say that we are applying the Y-procedure to y.

For the first bijection, $f : X_X \to Y_X$, the first thing that we have to check is that f does indeed map elements of X_X to Y_X as advertised. If $x \in X_X$, then that means applying the X-procedure to x fails at some step because we cannot apply g^{-1}. Now, x is mapped by f to the element $f(x)$, and consider what happens when we apply the Y-procedure to $f(x)$: first we apply $f^{-1}(f(x))$, which is clearly valid since $f(x)$ lies in the range of f. Next, we apply g^{-1} and f^{-1} alternately, and we see that the procedure for doing so *exactly mirrors* the steps we take when we apply the X-procedure to x. In other words, the X-procedure applied to x terminates because we cannot apply g^{-1}, *if and only if* the Y-procedure applied to $f(y)$ terminates because we cannot apply g^{-1}. This proves that $f(x) \in Y_X$.

Next, we must show that f is a bijection between these sets. Consider any point $y \in Y_X$. When we apply the Y-procedure to y, the first thing we do is attempt to apply f^{-1}. However, since $y \in Y_X$, the Y-procedure cannot stop here (we must terminate when we cannot apply g^{-1} anymore). This says that if $y \in Y_X$, then we can successfully apply f^{-1} to y, which is what we wanted to show. (We already know that f is injective. Saying that we can apply f^{-1} to any $y \in Y_X$ is saying that every $y \in Y_X$ lies in the range of f, so f is also surjective, so f is a bijection.)

We have shown the bijection between X_X and Y_X, and in principle, we must now prove the bijections between X_Y and Y_Y, and between X_∞ and Y_∞. The proofs end up looking almost identical to the one described above, so let's be brief: g^{-1} maps X_Y to Y_Y, because both x and $g^{-1}(x)$ must terminate when we are unable to apply f^{-1}; and this is a bijection because we know that if $x \in X_Y$, then we succeed at applying g^{-1}. Also, f maps X_∞ to Y_∞ because if the procedure never terminates for x, then it clearly never terminates for $f(x)$ either; and f is a bijection because if $y \in Y_\infty$, then because the Y-procedure applied to y never terminates, we must succeed at applying f^{-1}. □

Here is an example of what sort of bijection is constructed in the theorem.

Example 1. We will construct a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ (here, $X = Y = \mathbb{N}$) from the two injections $f(x) = 2x$ and $g(x) = 2x$. From the Cantor-Schröder-Bernstein Theorem, we know that the bijection will consist of $f(x) = 2x$ and $g^{-1}(x) = x/2$; it only remains to determine what X_X, X_Y, and X_∞ are.

1. Suppose that we take $x \in X$, and try to apply g^{-1} to x. If this fails, then we know that $x \in X_X$. This will fail exactly when x is odd, or equivalently, when $x \equiv 1 \pmod 2$.

2. Suppose that we successfully apply g^{-1} to x, which means that x is even. What happens when we fail to apply f^{-1}? That means that x is even, but $x/2$ is odd, which is to say that $x \equiv 2 \pmod 4$. Here, $x \in X_Y$.

3. Assume that the previous two steps succeeded, so that x is divisible by 4. If we cannot apply g^{-1} here, then that means x is even, $x/2$ is even, but $x/4$ is odd, which is equivalent to $x \equiv 4 \pmod 8$. Here, $x \in X_X$.

4. The pattern continues...

We can write the full bijection φ as follows: if $x \equiv n/2 \pmod n$, where n is some *even* power of 2 (such as 2^2, 2^4, 2^6, etc.), then we map x to $x/2$ (this is g^{-1}); otherwise, we map x to $2x$ (this is f).

Example 2. Can we find a bijection from $(0,1]$ to $[0,1]$? We supply two injections. The first injection, $f : (0,1] \to [0,1]$, is obvious. For the second injection, we can define $g(x) = x/2 + 1/4$. (Basically, we squash the interval $[0,1]$ to $[0,1/2]$, and then add a constant to prevent any input from mapping to 0.) The Cantor-Schröder-Bernstein Theorem supplies a bijection φ, so we know that the cardinalities of $(0,1]$ and $[0,1]$ are the same (in fact, they have the same cardinality as $\mathbb{R}$). However, we choose to skip the explicit construction of this bijection as we did in the example above.

For this specific question, we can give an explicit bijection directly. Since $\mathbb{Q} \cap [0,1]$ is countable, we can list the elements $\mathbb{Q} \cap [0,1] = \{q_0, q_1, q_2, q_3, \dots\}$, where we can take $q_0 = 0$. Then, define the bijection $f : [0,1] \to (0,1]$ by $f(q_i) := q_{i+1}$ for each $i \in \mathbb{N}$, and $f(x) := x$ for all $x \notin \mathbb{Q} \cap [0,1]$.