1 Bayes Rule – Man Speaks Truth

(a) A man speaks the truth 3 out of 4 times. He flips a biased coin that comes up Heads $1/3$ of the time and reports that it is Heads. What is the probability it is Heads?

(b) A man speaks the truth 3 out of 4 times. He rolls a fair 6-sided die. When you ask him if the die came up with a 6, he answers "yes". What is the probability it is really 6?

2 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears?

(c) A building has n floors numbered $1, 2, \dots, n$, plus a ground floor G. At the ground floor, m people get on the elevator together, and each gets off at a uniformly random one of the n floors (independently of everybody else). What is the expected number of floors the elevator stops at (not counting the ground floor)?

(d) A coin with heads probability p is flipped n times. A "run" is a maximal sequence of consecutive flips that are all the same. (Thus, for example, the sequence $HTHHHTTH$ with $n = 8$ has five runs.) Show that the expected number of runs is $1 + 2(n-1)p(1-p)$. Justify your calculation carefully.

3 Uniform Distribution

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0, 10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

4 High and Low States

Suppose that we have n "high" states $H_1, \ldots, H_n$ and n "low" states $L_1, \ldots, L_n$. The high state H_k has a probability p of transitioning to L_k, and a probability $1 - p$ of staying at H_k. The low state L_k has a probability q of transitioning to the next high state H_{k+1} (wrapping around, so L_n can transition to H_1), and a probability $1 - q$ of staying at the same location. Is the Markov chain aperiodic? What is the limiting distribution?