1 Seating Arrangement

N people have come to watch a play and were given a row with exactly N consecutive seats. They have decided on the following method for determining their seating arrangement. The first person chooses any seat in the row, then the next person has to sit next to the first. The third sits next to one of the first two and so on until all N are seated. In other words, no person can take a seat that separates them from the people who have already sat down. Prove that there are 2^{N-1} possible seating arrangements that can arise from this procedure. [*Hint*: Use induction.]

2 Triangular Faces

Suppose we have a connected planar graph G with v vertices and e edges such that $e = 3v - 6$. Prove that in any planar drawing of G, every face must be a triangle; that is, prove that every face must be incident to exactly three edges of G.

3 Simplifying Some "Little" Exponents

For the following problems, you must both calculate the answers and show your work.

(a) What is $7^{3,000,000,000} \mod 41$?

(b) What is $2^{2017} \bmod 11$?

(c) What is $2^{(5^{2017})} \bmod 11$?

4 Functions and Countability

For any non-empty set S, let D_S be the set of all functions $f : S \to \mathbb{N}$ and R_S be the set of all functions $f : \mathbb{N} \to S$.

(a) Under what conditions on S is D_S countable?

(b) Under what conditions on S is R_S countable?

5 Fixed Points

Consider the problem of determining if a function F has any fixed points; that is, we want to know if there is any input x such that $F(x)$ outputs x. Prove that this problem is undecidable.