CS 70 Discrete Mathematics and Probability Theory
Summer 2018 Sinho Chewi and Vrettos Moulos **DIS 2A**

1 Leaves in a Tree

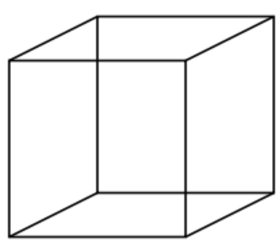
A *leaf* in a tree is a vertex with degree 1.

(a) Consider a tree with $n \geq 3$ vertices. What is the largest possible number of leaves the tree could have? Prove that this maximum m is possible to achieve, and further that there cannot exist a tree with more than m leaves.

(b) Prove that every tree on $n \geq 2$ vertices must have at least two leaves.

2 Cube Dual

(a) We define a graph G by letting the vertices be the corners of a cube and having edges connecting adjacent corners. Draw a planar representation of G and the corresponding dual planar graph. (Below is a picture of a cube, provided for reference)



(b) Find a spanning tree of your planar drawing and identify the corresponding spanning tree of the dual planar graph.

3 Planarity

Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

4 Planarity and Graph Complements

Let $G = (V, E)$ be a graph. We define the complement of G as $\overline{G} = (V, \overline{E})$ where $\overline{E} = (V \times V) - E$; that is, $\overline{G}$ has the same set of vertices as G, but an edge e exists is $\overline{G}$ if and only if it does not exist in G.

(a) Suppose G has v vertices and e edges. How many vertices and edges does $\overline{G}$ have?

(b) Prove that for any graph with at least 13 vertices, G being planar implies that $\overline{G}$ is non-planar.

(c) Is the converse of the previous part true? That is, if $\overline{G}$ is non-planar, does that imply that G is planar? Prove this or give a counterexample.